

A
TREATISE
OF
Conic Sections,

By ROBERT STEELL.



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THE
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DEDICATED
TO THE
AUTHOR



TO THE
LEARNED
THE
Provost, Fellows,
AND
SCHOLARS
OF THE
College of Dublin,
THIS
TREATISE
OF
Conic Sections,
Is HUMBLY
DEDICATED
BY THE
AUTHOR.

TO THE
F A R M E R S

Provoost, Fellows,

8 C H O A C R S

College of Dublin

T R E A T I S E


Conic Sections

D E D I C A T E D

A U T H O R



THE
PREFACE.

Y design in this Treatise, being to Demonstrate the principal Properties of the *Conic Sections*, in the most easie manner; I shall not confine my self strictly either to the *analytic*, or *Synthetic* Method, but shall use both indifferent-ly, as I shall find the one or the other best answer that End.
Neither

P R E F A C E.

Neither shall I scruple to borrow, or alter what I find for my purpose, in the Writings of others on the same Subject.

Mr. *Walton* on perusing the Manuscript, was pleas'd to Communicate some Properties which the Reader will find in the *Italian* Character.



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The Explication of *Signs* and
Characters, used in this Trea-
tise.

\therefore	} Signifies	Therefore.
\parallel		Parallel.
\angle		Angle.
L		Right Angle.
\square		Parallelogram.
\triangle		Triangle.
$\overline{a+b}^2$		Square of $a + b$.
\cup		Difference, or Excess.
$\sqrt[3]{t}$		SquareRoot of the Cube of t .
$b :: c. d$		Proportionality.



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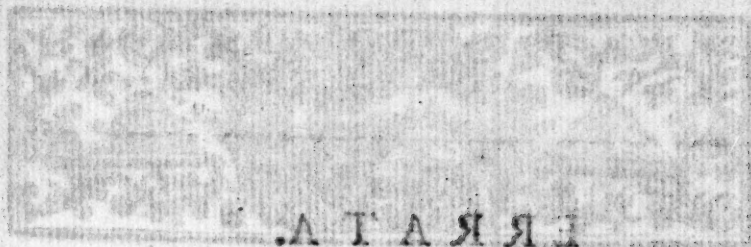
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P. I.	For	Read
7	21	Ordinate FG
32	7	AK
37	24	AK
40	19	$2xyX = X + 2$
46	4	YT
50	14	$2xyX = X + 2$
53	4	$2xyX = X + 2$
58	20	$2xyX = X + 2$
91	2	D
96	13	NS
	14	TO
	15	VP

In the PLATES

Plate I. of the Parabola, Fig. 6. where F
and S V intersect, place O.
Plate III. of the Hyperbola, Fig. 14. where
A I intersect V, place D.
Ibid. Fig. 15. for w, read e.

ERRATA.

P.	L.	For	Read
7	21	Ordinate. FG	Ordinate FG.
35	7	Ak	AK
37	24	$\frac{1}{2}KH$	$\frac{1}{2}KX$
40	19	$g \times tX - X^2 + g^2$	$g \times tX - X^2 + g^2$
46	4	YF	YT
50	14	$t^2 c^2$	$t^4. c^2$
53	4	$c + D$	$c + d$
78	20	$\sqrt{p} = p$	$\sqrt{p} = b$
91	5	$\frac{1}{2}PD^2$	$\frac{1}{4}D^2$
96	13	NS $\left. \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right\} = MR$	NS $\left. \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right\} = MR$
	14	TO $\left. \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right\} = MR$	TO $\left. \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right\} = MR$
	15	VP $\left. \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right\} = MR$	VP $\left. \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right\} = MR$

In the *PLATES*.

Plate I. of the Parabola, Fig. 6. where FT and SV intersect, place O.

Plate III. of the Hyperbola, Fig. 14. where AL intersects Vx, place D.

Ibid. Fig. 15, for *n*, read *a*.

Corrected



Conic Sections.

PART I.

Of the PARABOLA.

The GENESIS.



From a Point V, in any indefinite right Line, there be taken $VD = VK$, and, from the point K as a Center, with the distance DG, you intersect CM, Perpendicular to DG, in the points C and M, those points will be in the Curve of a PARABOLA. Fig. I.

DEFINITIONS.

1. The Point V is the Vertex, and K the Focus of the PARABOLA.
2. The right Line DG, passing through the Focus is call'd the Axis.
3. A right Line Perpendicular to the Axis, and terminated by the Curve, is an Ordinate to the Axis, as GC.

A

4. The

Of the PARABOLA. PART I.

4. The distance (in the Axis) from the Vertex to the intersection of the Ordinate, is called the Abscissa of that Ordinate, as V G.

5. A right Line drawn from any point of the Curve, and Parallel to the Axe is called a Diameter as C Y; and the point in the Curve, from which it is drawn, is called the Vertex of that Diameter.

PROP. I.

Fig. I. **T**HE Square of any Ordinate, is equal to the Rectangle of the Abscissa of that Ordinate into Quadruple the distance of the Focus from the Vertex, that is, $GCq = VG \times 4KV$.

DEMONST.

Put $KV = VD = q$, $VG = x$, and $GC = y$, then, (by the Genesis) $GK = q \cup x$ and $DG = CK = q + x$; but (by 47 E. I.) $KC, q = KG, q + GC, q$, that is, $q^2 + 2qx + x^2 = q^2 - 2qx + x^2 + y^2$, or $4qx = y^2$. *i.e.* $GCq = VG \times 4KV$. Q. E. D.

COR. I.

The Squares of the Ordinates are to each other as their Abscissas. because $Y^2 = 4qX$, and $y^2 = 4qx$, therefore $Y^2. y^2 :: (4qX. 4qx ::) X. x$.

Definition, The Quadruple of the Focal distance is called the Parameter of the Axis, and is a third Proportional to any Abscissa and it's Ordinate. For by putting $4q = p$ it will be $pX = y^2$ therefore $x. y :: y. p$.

COR.

PART I. Of the PARABOLA.

COR. II.

The Ordinate, which passes through the Focus, is equal to half the Parameter of the Axe. For, in that case, $x=q$, therefore (by the Proposition) $4q^2=y^2$ and $y=(2q)^{\frac{1}{2}}p$.

PROP. II.

AS the Parameter of the Axe is to the sum Fig. II. of any two Ordinates, so is their difference to the difference of their Abscissas; that is, p . IN :: NO. NC.

DEMONST.

Put $HO=Y$, $GC=y$, $VH=X$, and $VG=x$; then (by Prop. I.) $pX=Y^2$, and $px=y^2$, therefore $pX-px=Y^2-y^2$, and $p.Y+y :: Y-y.X-x$; that is, p . IN :: NO. NC. Q. E. D.

PROP. III.

IF, from the Vertex a right Line be drawn so Fig. III. as to cut the Curve, and continued till it cut any Ordinate produc'd, it will be, as the Parameter of the Axe, is to the Ordinate drawn from the Intersection with the Curve, so is the produc'd Ordinate to its Abscissa, that is, p . GM :: HS. HV.

DEMONST.

Let $HS=b$, $VG=x$, $VH=X$, $HO=Y$, and $GM=y$. Then (by Prop. I.) $y^2.Y^2 :: (x$.

Of the PARABOLA. PART I.

($x. X ::$ by Similar Δ 's) $y. b$; therefore $\frac{b y y}{y}$
 $= (Y^2 =) pX$, or $b y = pX$; that is, $p. y ::$
 $b. X$; or, $p. GM :: HS. HV. \text{Q. } E. D.$

PROP. IV.

IF, from any point D in the Axe produced, a right Line be drawn intersecting the Curve in two points C and I , and the Ordinates CP , IH be drawn from the said Intersections; VD , will be a mean Proportional between VP and VH .

DEMONST

Put $VD = b$, $VP = x$, $VH = X$, then $PD = b + x$ and $HD = b + X$; but (from Similar Δ 's) $PDq. HDq :: PCq. IHq ::$ (by I.) $PV. HV$, that is, $b^2 + 2bx + x^2. b^2 + 2bX + X^2 :: x. X$. therefore $X - x \times b^2 = X - x \times Xx$; or $b^2 = Xx$, that is, $x. b :: b. X$, or $VP. VD :: VD. HV. \text{Q. } E. D.$

PROP. V.

IF any right Line touch the Curve, and an Ordinate be drawn from the point of Contact, then, I say, the Abscissa of that Ordinate shall be equal to the distance (in the Axe produc'd) from the Vertex of the Curve to the intersection
 Fig. IV. of the Tangent. that is, $GV = VT$.

DEMONST.

Let rF be an indefinitely small part of the Curve, and continued to T , draw $rs ||$ to the Ordinate,

PART I. Of the PARABOLA.

5.

Ordinate, and $Fp \parallel$ to the Axe, and let $Fp = n$, $rp = m$, $VT = a$; and the other Symbols as usual. then $Vs = x + n$, and $rs = y + m$; and by Similar Δ 's $m.n :: y.x + a$; therefore

$n \times \frac{y}{m} = \left(\frac{ny}{m} = \right) x + a$, and (by Prop. I.) $p \times Vs = sr, q$; and $p \times VG = GFq$; that is, $pn + px = y^2 + 2ym + m^2$, and $px = y^2$, therefore $y^2 + 2ym - pn = (px =) y^2$; that is, $n = \frac{2ym}{p}$; and consequently, $x + a = \left(\frac{2ym}{p} \times \frac{y}{m} = \frac{2y^2}{p} = \frac{2px}{p} = \right) 2x$. and $a = x$, or, $VT = GV$. Q. E. D.

PROP. VI.

IF, from the point of Contact, a right Line be drawn to the Focus, it shall be equal to the distance (in the Axe produc'd) from the Focus to the intersection of the Tangent, that is, KF Fig. IV. $= KT$.

DEMONST.

By Prop 5. $GT = 2x$, and (by Prop. I.) $KG = x - \frac{1}{2}p$, $\therefore KT = (GT - KG = 2x - x + \frac{1}{2}p = x + \frac{1}{2}p =$ (by the Genesis) KF . Q. E. D.

PROP. VII.

IF, to the Tangent, from the point of Contact, a Perpendicular be drawn, and produced to cut the Axe, then the distance in the Axe from that point, to the Ordinate drawn from the point of

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Of the PARABOLA. PART I.

of Contact, that is the *Subnormal* is equal to
 Fig. V. half the Parameter of the Axe. that is, $QG = \frac{1}{2}p$.

DEMONST.

For QG put b , then (by 8. Eu. 6) $GT.GC :: GC.GQ$; that is, $2x.y :: y.b \therefore 2xb = (y^2 =) px$, and $b = \frac{1}{2}p$, or, $QG = \frac{1}{2}p$. Q. E. D.

PROP. VIII.

THE distances from the Focus to the point of Contact, from the Focus to the intersection of the Tangent with the Axe, and from the Focus to the end of the Subnormal are equal. that is, $FC = FT = FQ$.

DEMONST.

Fig. V. By the Genesis $GF = x - \frac{1}{2}p$, and (by the 7th.) $QG = \frac{1}{2}p \therefore FQ = (GF + GQ = x + \frac{1}{2}p = (\text{by the Genesis}) FC = (\text{by the 6th.}) FT$. Q. E. D.

COROL. I.

Hence F is the Centre of a Circle passing through Q , C , and T .

COROL. II.

The Angle formed by the Tangent and Axe is equal to half the Angle formed by the Axe, and a straight Line drawn from the Focus to the point of Contact; that is, $\angle CTQ = \frac{1}{2} \angle CFQ$, by 31. Eu. 3.

PROP.

PART I. Of the PARABOLA.

PROP. IX.

IF, from the Vertex, a right Line be drawn Parallel to an Ordinate drawn from the point of Contact, and cut the Tangent, the Square of that Line shall be equal to the Rectangle of half the Parameter of the Axe into half the Abscissa of that Ordinate. that is, $VRq = \frac{1}{2}p \times \frac{1}{2}GV$.

DEMONST.

Let $VR = b$, the Δ 's TVR , TGC are Similar, but $VT = \frac{1}{2}GT$. by the 5. $\therefore VR = \frac{1}{2}GC$; that is, $b = \frac{1}{2}y \therefore b^2 = (\frac{1}{2}y)^2 =$ by Prop. 1. $\frac{1}{4}p \times \frac{1}{2}x$, or $VRq = \frac{1}{2}p \times \frac{1}{2}GV$. *Q. E. D.*

PROP. X.

IF, to the Tangent drawn to the Vertex of any Diameter a right Line be drawn Parallel, *Fig. VI.* the part of that Line which lies within the Curve, shall be Bisected by the Diameter, that is, the Ordinate $xb = bz$.

DEMONST.

Produce the Diameter Yb , and draw KR , VS Parallel to the Ordinate FG . then,

1. ΔGFT or $\square GS. \Delta KzP :: (GFq. Kzq :: GV. KV. ::$ by 1. E. 6) $\square GS. \square KS. \therefore \Delta KzP = \square KS.$

2. ΔGFT or $\square GS. \Delta HxP :: (GFq. Hxq :: GV. HV. ::) \square GS. \square HS. \therefore \Delta HxP = \square HS.$ but $\Delta HxP = \Delta KzP = \square HS = \square KS. i.e.$ the Figure $Hxzk = \square HR$, from which taking
ing

ing the common Figure $HYbzK$, there remains the $\triangle Yxb =$ and Similar to the $\triangle bRz$, and consequently $xb = bz$. *Q. E. D.*

COROL.

The Figure $bFTP = \triangle Yxb$, because $\triangle GFT = \square GS$, therefore the Figure $HYFT = (\square HF + \triangle GFT = \square HF + \square GS = \square HS =) \triangle HxP$. from which taking the common Figure $HYbP$, there remains the Figure $bFTP = \triangle Yxb$.

LEMMA.

Fig. XI. If FT be \parallel to bp , and the $\triangle brz =$ Trapezium $FbTp$, then $\overline{FT + bp} \times Fb = br \times zb$. because (by Hypothesis) $\overline{FT + bp} \times p = zb \times q \therefore FT + bp. zb :: q. p. ::$ (by Similar \triangle 's) $rb. bF$. and $\overline{FT + bp} \times Fb = zb \times br$. *Q. E. D.*

Fig. VI. *Definition*, Let $FS. FO :: 2FT$. P the Parameter belonging to the Diameter FY . then,

PROP. XI.

THE Rectangle of the Parameter (so obtained) into any Abscissa of that Diameter, is equal to the Square of the Ordinate of that Ab-
Fig. VI. scissa. that is, $P \times Fb = xb, q = bz, q$.

DEMONST.

By the Definition $\frac{P}{2FT} = \frac{FO}{FS} =$ by Similar \triangle 's, $\frac{bx}{bY}$; and (by the preceding Lemma) $2FT$

x

PART I. Of the PARABOLA.

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$$\times Fb = yb \times bx \therefore \left(\frac{P}{2FT} \times 2FT \times bF = \frac{bx}{by} \right. \\ \left. \times by \times bx \right) \text{ or } P \times Fb = bxq = bzq. \text{ Q.E.D.}$$

PROP. XII.

THE Parameter of any Diameter is equal to the Paramater of the Axe added to Quadruple the Abscissa of the Ordinate drawn from the Vertex of that Diameter. *i. e.* $P = p + 4GV$. Fig. VII.

DEMONST.

From the Vertex draw Vb Parallel to the Tangent FT , which (by the 10th.) will be an Ordinate to the Diameter FY . then, by reason of Parallels $bF = VT =$ (by the 5) $GV = x$. and by the last, $Px = (bVq = FTq = FGq + GTq = 4x^2 + y^2 =) 4x^2 + px \therefore P = (4x + p \text{ or }) p + 4GV$. Q. E. D.

PROP. XIII.

THE distance from the Focus to the Vertex of any Diameter is equal to one fourth of the Parameter of that Diameter. that is, $KF = \frac{1}{4}P$.

DEMONST.

By the last, $P = p + 4VT$, and (by the first) $p = 4KV \therefore P = 4KV + 4VT$; and $\frac{1}{4}P = (KV + VT = KT =$ by the 6.) KF . Q. E. D.

B

PROP.

ing the common Figure $HYbzK$, there remains the $\triangle Yxb =$ and Similar to the $\triangle bRz$, and consequently $xb = bz$. *Q. E. D.*

COROL.

The Figure $bFTP = \triangle Yxb$, because $\triangle GFT = \square GS$, therefore the Figure $HYFT = (\square HF + \triangle GFT = \square HF + \square GS = \square HS =) \triangle HxP$. from which taking the common Figure $HYbP$, there remains the Figure $bFTP = \triangle Yxb$.

LEMMA.

Fig. XI. If FT be \parallel to bp , and the $\triangle brz =$ Trapezium $FbTp$, then $\overline{FT} + \overline{bp} \times \overline{Fb} = \overline{br} \times \overline{zb}$. because (by Hypothesis) $\overline{FT} + \overline{bp} \times \overline{p} = \overline{zb} \times \overline{q} \therefore \overline{FT} + \overline{bp} \times \overline{zb} :: \overline{q} \times \overline{p} ::$ (by Similar \triangle 's) $\overline{rb} \times \overline{bF}$. and $\overline{FT} + \overline{bp} \times \overline{Fb} = \overline{zb} \times \overline{br}$. *Q. E. D.*

Fig. VI. Definition, Let $FS : FO :: 2FT$. P the Parameter belonging to the Diameter FY . then,

PROP. XI.

THE Rectangle of the Parameter (so obtained) into any Abscissa of that Diameter, is equal to the Square of the Ordinate of that Abscissa. that is, $P \times Fb = xb, q = bz, q$.

DEMONST.

By the Definition $\frac{P}{2FT} = \frac{FO}{FS} =$ by Similar \triangle 's, $\frac{bx}{bY}$; and (by the preceding Lemma) $2FT$

x

PART I. Of the PARABOLA.

9

$$\times Fb = yb \times bx \therefore \left(\frac{P}{2FT} \times 2FT \times bF = \frac{bx}{by} \right. \\ \left. \times by \times bx \right) \text{ or } P \times Fb = bxq = bzq. \text{ Q. E. D.}$$

PROP. XII.

THE Parameter of any Diameter is equal to the Parameter of the Axe added to Quadruple the Abscissa of the Ordinate drawn from the Vertex of that Diameter. *i. e.* $P = p + 4GV$. Fig. VII.

DEMONST.

From the Vertex draw Vb Parallel to the Tangent FT , which (by the 10th.) will be an Ordinate to the Diameter FY . then, by reason of Parallels $bF = VT =$ (by the 5) $GV = x$. and by the last, $Px = (bVq = FTq = FGq + GTq = 4x^2 + y^2 =) 4x^2 + px \therefore P = (4x + p \text{ or }) p + 4GV$. Q. E. D.

PROP. XIII.

THE distance from the Focus to the Vertex of any Diameter is equal to one fourth of the Parameter of that Diameter. that is, $KF = \frac{1}{4}P$.

DEMONST.

By the last, $P = p + 4VT$, and (by the first) $p = 4KV \therefore P = 4KV + 4VT$; and $\frac{1}{4}P = (KV + VT = KT = \text{by the 6.}) KF$. Q. E. D.

B

PROP.

PROP. XIV.

IF, from the Focus, a Perpendicular be drawn to any Tangent; then the Square of that Line shall be equal to the Rectangle under the Focal Distance, and the Distance of the point of Contact from the Focus. *i. e.* $KOq = KV \times KF$.

DEMONST.

From the Vertex draw $VO \parallel$ to GF , which will Coincide with the point O , because (by the 5th.) $GV = VT \therefore$ (by 2. E. 6.) $TO = OF$, and because $\angle KOT$ is right \therefore (by the 8. E. 6) $TK \cdot KO :: KO \cdot KV$. and $KOq = (TK \times KV) FK \times KV$. *Q. E. D.*

PROP. XV.

IF an Ordinate to any Diameter pass through the Focus, then the Abscissa of that Ordinate shall be equal to one fourth, and the Ordinate equal to one half of the Parameter of that Diameter.

DEMONST.

1. From Parallels, $bF = (KT = \text{by 6. } KF = \text{Fig VII. by 13) } \frac{1}{2}P$.
2. Since $bF = \frac{1}{2}P$ and (by the 11) $P \times bF = bCq \therefore \frac{1}{4}P^2 = \overline{bC}^2$ and $\frac{1}{2}P = bC$. *Q. E. D.*

PROP.

PART I. Of the PARABOLA.

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PROP. XVI.

THE distance (in the Axe) from the intersection of the Tangent, to the end of the Subnormal, is equal to half the Parameter of that Diameter, whose Vertex is the point of Contact. that is, $QT = \frac{1}{2}P$.

DEMONST.

By the 13. $FK = \frac{1}{4}P$; and (by 8) $FK = QK = KT \therefore QT = (QK + KT = \frac{1}{4}P + \frac{1}{4}P = \frac{1}{2}P$. *Q. E. D.*

PROP. XVII.

IF a double Ordinate be drawn from the Point of Contact, and another double Ordinate be drawn below; and cut the Tangent produc'd. Then as the double Ordinate passing through the Point of Contact, is to the Sum of the two Ordinates, so is their difference, to the external part of the lower Ordinate added to the difference of the Ordinates. that is $MF. OL :: IL. BL$.

Fig.
VIII.

DEMONST.

Let $VG = x$, then (by 5) $GT = 2x$; $FG = y$, $OL = c$, $IL = m$, and $LB = d$. Then (by 2.) $p. c :: m. \frac{mc}{p} = LF$, and from Similar Δ 's, $2x. y :: \frac{mc}{p}. d \therefore 2pxd = ymc$ and (because $px = y^2$) $2yd = mc$, or $2y. c :: m. d$. i.e. $MF. OL :: IL. BL$. *Q. E. D.* PROP.

PROP. XVIII.

Fig. VIII. THE same things being suppos'd as before; the difference of the Ordinates is a mean Proportional between the Double of the upper Ordinate, and the External part of the Lower, *i. e.* FM. IL :: IL. BI.

DEMONST.

For BI put c , IL, m , and FM, $2y$, then OL $= 2y + m$, and (by 17.) $2y. 2y + m :: m. d \therefore d = \frac{2ym + m^2}{2y}$; and, $c = \left(d - m = \frac{2ym + m^2}{2y} - m = \right) \frac{m^2}{2y}$; that is, FM. IL :: IL. BI. Q. E. D.

PROP. XIX.

THE same things being still supposed; as the Double of the lower Ordinate added to the External part, is to the Sum of the two Ordinates, so is the External part of the lower Ordinate added to the difference of the Ordinates, to the difference of the Ordinates. that is, OB, LB :: OL. IL.

DEMONST.

Let OL $= c$, LB $= d$, IL $= m$; then OB $= c + d$ and MF $= c - m$. But (by 17) MF. OL :: IL. BL; that is, $c - m. c :: m. d \therefore cd - dm = cm$; and $cm + dm = cd$; or $c + d. c :: d. m$; that is, OB, LB :: OL. IL. Q. E. D.

PROP.

PROP. XX.

STILL supposing the same things; having OI, and BI given; 'tis requir'd to find IL.

Let $KL = b$, $IL = m$, $BI = a$, and $OI = c$, then (by 18) $KL \cdot (MF) IL :: IL \cdot BI$, that is,

$$b \cdot m :: m \cdot a :: ba = m^2 \text{ and } b = \frac{m^2}{a} \text{ also, } c =$$

$$\left(b + 2m = \frac{m^2}{a} + 2m = \right) \frac{m^2 + 2am}{a} \therefore ac = m^2$$

$$+ 2am. \text{ Whence } m = \sqrt{a + c \times a} - a.$$

COR.

Hence from a point B without the Curve (and not in the Axe produc'd) we may draw a Tangent. For, if, from the given point, we draw IO Perpendicular to the Axe, and then find a mean proportional between OB and IB, from which if we take IB, and set the remainder from I, to L; and then from L, draw LF, Parallel to the Axe, the point F is determin'd, to which, if from the given point B, a right line be drawn, it will touch the Curve. Fig. VIII.

PROP. XXI.

IF FP touch the Curve in F, and, from any Points M, S, in that Tangent the right lines BM, SD, be drawn || to the Axe and cut the Ordinate in B and D; then $MO \cdot FBq :: SR \cdot FDq$. Fig. IX.

DEMONST.

DEMONST.

Let $MO = b$, $FB = c$, $SR = d$, and $FD = q$,
 also $GV = VT = x$. Then by II. $x. b :: (FTq. FMq ::$ by Similar Δ 's) $y^2. c^2$ and $x. d :: (FTq. FSq ::$ by Similar Δ 's) $y^2. q^2. \therefore$ (by Equality) $b. c^2 :: d. q^2$, or $MO.FBq :: SR.FDq$. *Q. E. D.*

PROP. XXII.

IF, from any Point in the Tangent, a right line be drawn Parallel to the Axe, and cut an Ordinate, the Rectangle of the Parameter of the Axe into the External part of that line, is equal to the Square of the Segment of the Ordinate intercepted between that line and the point of Contact. that is, $p \times MO = FBq$, or $p \times RS = FDq$.

DEMONST.

By the last $\frac{c^2 x}{b} = (y^2 = \text{by the 1st.}) px$;
 also $\frac{q^2 x}{d} = (y^2 =) px \therefore \overline{pb}^2 = c^2$ and $dp = q^2$. *i. e.* $p \times MO = FBq$ and $p \times RS = FDq$.
Q. E. D.

PROP. XXIII.

Fig. 1X. **I**F FP touch the Parabola in F, and if, from any point S, in the Tangent a right line SD, be drawn Parallel to the Axe, and cut another right line FC drawn from the Point of Contact any how within the Curve; then the Curve shall cut

PART I. Of the PARABOLA.

15

cut the first line in the same Proportion that the first line cuts the second; that is, $SR.RD :: FD.DC$.

DEMONST.

Draw PC Parallel to SD , and let $CP = r$, $RS = c$, $FS = d$, $RD = p$, $PS = m$, $FD = g$, and $DC = b$. Then $c.r :: (d^2. d+m)^2 ::$ by Similar Δ 's) $g^2. g+b)^2$ and (by Similar Δ 's.) $r.g+b :: c+p.g$, therefore $\frac{cg^2 + 2cgb + cb^2}{g^2} = (r =)$ $\frac{cg + cb + pg + pb}{g}$; and $cgb + cb^2 = pg^2 + pgb$; and dividing by $g+b$; $cb = pg$, or, $c.p :: g.b$; that is, $SR.RD :: FD.DC$. $\mathcal{Q}. E. D.$

PROP. XXIV.

AS the Abscissa is to the Square of the Ordinate, so is any right line drawn within the Curve, and Parallel to the Axe, to the Rectangle of the Parts of the Ordinate which it divides. that is, $VG.FGq :: OB.FB \times BC$. Fig. IX.

DEMONST.

Let $OB = m$, $FB = c$, $BC = r$, $MO = b$, then (by 21) $x.b :: y^2.c^2$ and (by the last) $b.m :: c.r :: \frac{c^2x}{y^2} = (b =) \frac{mc}{r}$; and $rcx = my^2$. or $x.y^2 :: m.rc$. that is, $VG.FGq :: OB.FB \times BC$. $\mathcal{Q}. E. D.$

COROL.

COROL.

OB. $FB \times BC :: RD. FD \times DC$. Because (by this Prop.) $OB. FB \times BC :: (VG. FGq ::) RD. FD \times DC$.

PROP. XXV.

IF a Tangent cut any Diameter produced, and if, from the point of Contact, an Ordinate be drawn to that Diameter; then the distance (in the Diameter produced) between the Vertex and Intersection of the Tangent, shall be equal to the Abscissa of the Ordinate; that is, $RS = SO$.

Fig. X.

DEMONST.

Let $OS = x$, $Cr = OP = n$, $tr = m$, $RS = a$, $OC = y$, and then Pt , (which is supposed to be indefinitely near to OC) will be $= y + m$. and $SP = x + n$. then by Similar Δ 's $m. n :: y$. $x + a \therefore x + a = n \times \frac{y}{m}$ and by the II. $p \times SO = OCq$, also $p \times SP = Ptq$. i.e. $px = y^2$, and $px + pn = y^2 + 2ym + m^2$; whence $px = y^2 + 2ym - pn \therefore y^2 = y^2 + 2ym - pn$, or $2ym = pn$; and $n = \frac{2ym}{p}$; also (by Substitution from the first Equation) $x + a = \left(\frac{2ym}{p} \times \frac{y}{m} = \frac{2y^2}{p} = \frac{2px}{p} = \right) 2x$; and $a = (2x - x) = x$; or, $RS = SO$. Q. E. D.

PROP.

PROP. XXVI.

IF a Diameter be drawn from the Intersection of any two Tangents, it will Bisect the line which joins the points of Contact.

DEMONST.

From the points of Contact (Y, C) draw the Ordinates $Y\phi, C\phi$; then by the last $RS = SO$ and $RS = S\phi \therefore SO = S\phi$, and consequently $Y\phi$ and $C\phi$ being Ordinates to the same Diameter and Abscissa are equal, and in the same right line. *Q. E. D.*

COROL.

Hence we have another method of drawing Tangents to the Parabola from any point without the Curve. For, if from the given point (as R) you draw a Diameter (as RP) and in that Diameter set, from the Vertex, the Abscissa (SO) equal to the External part (RS) and then through the extremity of the Abscissa (O) drawing a right line Parallel to the Tangent (xy) at the Vertex of that Diameter, the Extremities of that line (as C, Y) will be the points in the Curve in which lines drawn from the given point will touch it.

PROP. XXVII.

IF, from the Extremity of any Ordinate ($x\phi$) to a Diameter, a right line (as xY) be drawn at right Angles to the Diameter; then the distance

Fig. VI.

C

tance

tance (xT) in that line from the extremity of the Ordinate to the Diameter, shall be a mean proportional between the Parameter of the Axe, and the Abscissa of that Ordinate. that is, $p \cdot xT :: xT \cdot Fb$.

DEMONST.

Put $xY = a$, $Fb = X$, then (by 1.) $FGq = px$, and (by 5) $GTq = 4x^2 \therefore$ (by 47. E. 1) $FTq = px + 4x^2 = p + 4x \times x$. But (by 12) $xbq = p + 4x \times X$. and from Similar Δ 's $FTq \cdot FGq :: xbq \cdot xYq$. that is, $p + 4x \times x \cdot px :: p + 4x \times X \cdot a^2 \therefore a^2 = pX$, and $p \cdot a :: a \cdot X$ or $p \cdot xY :: xY \cdot Fb$. Q. E. D.



Parabola.

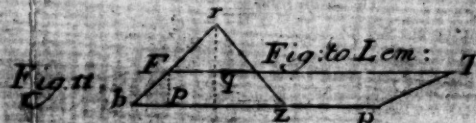
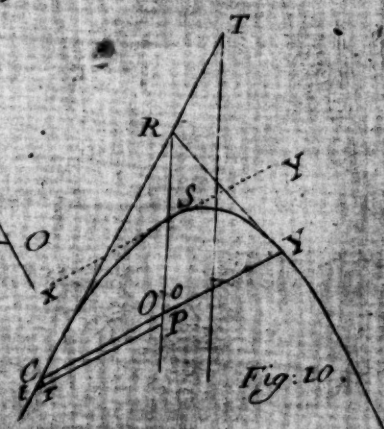
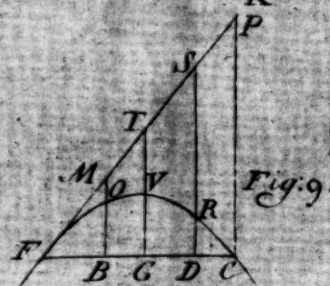
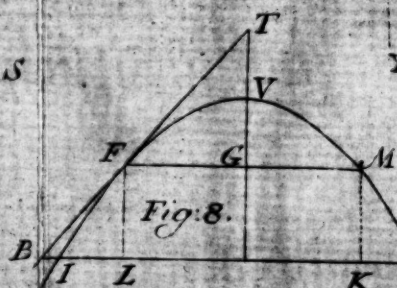
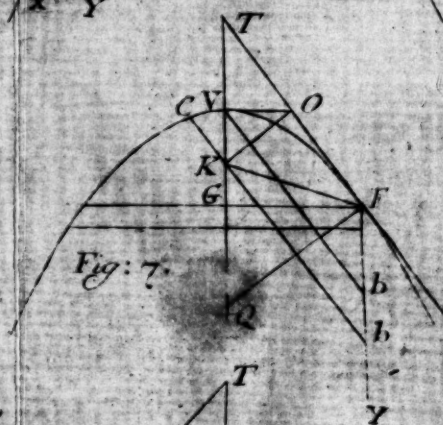
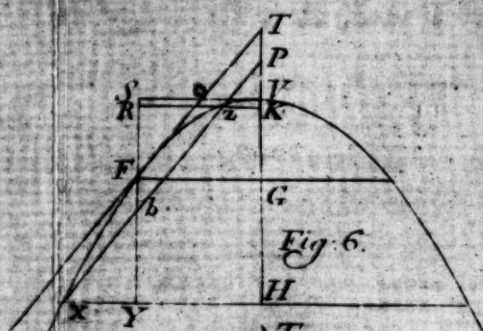
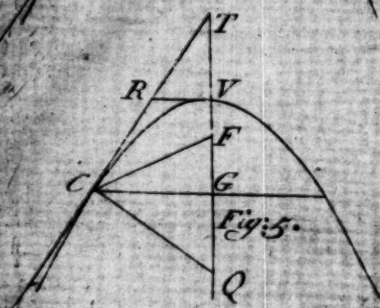
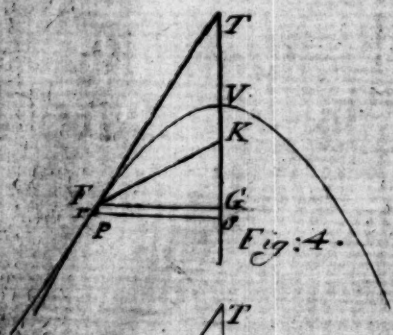
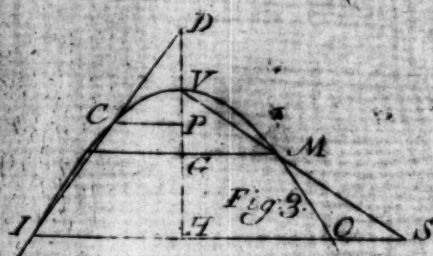
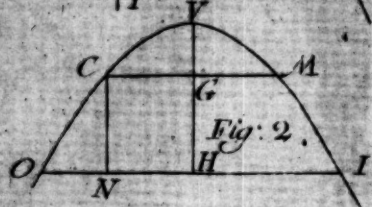
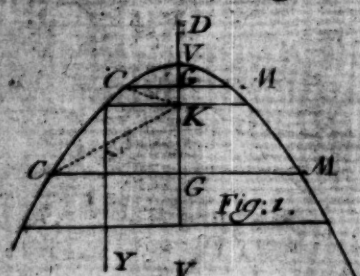


Plate 1.



Conic Sections.

PART II.

Of the ELLIPSE.

The GENESIS.



F, upon a Plane, any right Fig. I.

Line be taken, as AB, and
 $BH = AK$, and, the point
 G be taken any where in
 that Line; Then, if, with
 the Radius AG, from the
 point K, you describe an Arc
 at F, and with the Radius

GB from the Centre H, you Intersect the former Arc, and if, from the Points H and K, you draw the Lines HF, KF, I say $HF + KF = AB$. For (by Construction) $KF = AG$, and $HF = GB$ $\therefore HF + KF = (AG + GB =) AB$. In like manner an indefinite Number of
 Points

Points may be found, and if a Curve-Line be drawn through them it is called an ELLIPSE.

DEFINITIONS.

Fig. II.

1. The Points H, and K are called the Foci.
2. A Diameter is a right-Line which passes through C, the middle of A B, and Biseects all Lines within the Curve, that are Parallel to the Tangent which touches it's Vertex, and the Lines so Biseected are called Ordinates to that Diameter, so F Y, is a Diameter: $xo = oz$, are Ordinates, being Parallel to the Tangent which touches the Curve in F, the Vertex of the Diameter.
3. The Point of Concourse (C) of all the Diameters is called the Center.
4. That Diameter to which the Ordinates stand at right-Angles is called the Transverse Axe, as A B; and that which passes through the Center, and cuts it at right-Angles, is called the Conjugate Axe; as E D.
5. The Point, where the Ordinate Intersects the Diameter, is called the Point of Application; as G, and o.
6. The Segment of the Diameter intercepted between the Vertex and Point of Application, is called the Abscissa; as F o, o Y; or B G, A G.

PROP. I.

Fig. III. **A**S the Square of any Ordinate to the Transverse Axe, is to the Rectangle of the Abscissas which it divides; so is the Square of the Conjugate, to the Square of the Transverse Axe.

DEMONST.

DEMONST.

Let $AB = t$, $DE = c$, $KC = b$, $CG = x$, $FG = y$, and $FH = z$; then $KH = 2b$, and $KG = b + x$, $GH = b \cup x$, and (by the Genesis) $KF = t - z$, the Points K and H , being the Foci. then $KEq - ECq = KCq$, that is, $\frac{1}{4}t^2 - \frac{1}{4}c^2 = b^2$, by 47. Eu. 1. and (by the 13. and 12. E. 2.) $KFq = FHq + KHq \mp 2KH \times HG$. i. e. $t^2 - 2tz + z^2 = z^2 + 4b^2 + 4bx - 4b^2$; $\therefore z = \frac{t^2 - 4bx}{2t}$ and by Squaring both sides, $\frac{t^4 - 8t^2bx + 16b^2x^2}{4t^2} = (z^2 = FG^2 + GH^2 =) y^2 + b^2 - 2bx + x^2$; which being clear'd of fractions and contradictory terms will be $t^4 + 16b^2x^2 = 4t^2y^2 + 4t^2b^2 + 4t^2x^2$, and Substituting, for $16b^2$ and $4b^2$ in this Equation, their respective values in the first, and throwing out contradictory terms, and, dividing by 4, we shall have $t^2y^2 = \frac{1}{4}t^2c^2 - c^2x^2$; which reduced to an Analogy gives $y^2. \frac{1}{4}t + x \times \frac{1}{4}t - x :: c^2. t^2$ i. e. $FGq. AG \times GB :: DE^2. AB^2$. Q. E. D.

COR.

Let any Abscissa be x , and its Ordinate y , the Transverse Axis, t , and the Conjugate c , (which Symbols represent the same things in all the following Demonstrations) then by this Theorem, $t^2. c^2 :: t - x \times x. y^2$, or $t^2y^2 = c^2tx - c^2x^2$; which I call the Equation of the Curve.

Definition. A third Proportional to the Transverse and Conjugate Axes is called the Parameter

parameter of the Axe; that is, if you put p , for the Parameter, then $t.c :: c.p. \therefore tp = c^2$.

PROP. II.

AS the Transverse Axe, is to its Parameter, so is the Rectangle of any two Abscissas; to the Square of the Ordinate which divides them.

DEMONST.

By the construction of the Parameter $tp = c^2$, and by putting tp in the Equation of the Curve for c^2 , we shall have a new Equation of the Curve in the Terms of the Parameter &c. viz. $ty^2 = tpx - px^2$, or $y^2 = \frac{p}{t} \times tx - x^2 \therefore t.p :: t - x \times x.y^2$. Q. E. D.

COROL.

As the Rectangle of any two Abscissas, is to the Square of the Ordinate which divides them; so is the Rectangle of any other two Abscissas, to the Square of the Ordinate which divides them: For, (by this Prop.) $t - x \times x.y^2 :: (t.p ::) t - X \times X.Y^2$.

PROP. III.

THE Transverse Axe into one fourth of its Parameter, is equal to the Rectangle of the greatest and least distance of either Focus from the Vertex; that is, $\frac{1}{4}p \times AB = AH \times HB = BK \times KA$.

DEMONST.

DEMONST.

Let $HB = q$, then, $HA = t - q$, and $CH = \frac{1}{2}t - q$. But $HEq = ECq + CHq$. *i. e.* $\frac{1}{4}t^2 =$ Fig. III.
 $\frac{1}{4}t^2 - tq + q^2 + \frac{1}{4}c^2$, or $t - q \times q = (\frac{1}{4}c^2 =)$
 $\frac{1}{4}tp$. or $\frac{1}{4}p \times AB = AH \times HB$. Q. E. D.

COROL.

The Semi-conjugate Axe is a mean Proportional between the greatest and least distance of either Focus from the Vertexes: for since $t - q \times q = \frac{1}{4}c^2 \therefore t - q : \frac{1}{2}c :: \frac{1}{2}c : q$. that is, $AH.CD :: CD.HB$.

PROP. IV.

THE Parameter of the Axe is double the Ordinate applied to the Focus.

DEMONST.

Let the Focal distance be q , and the Ordinate passing through the Focus y , then (by the 2 d.) Fig. IV.
 $t.p :: t - q \times q.y^2$. But (by Prop. 3 d.) $t - q \times q = \frac{1}{4}tp \therefore t.p :: \frac{1}{4}tp$. ($\frac{1}{4}p^2 =$) y^2 , and $\frac{1}{2}p = y$, or $p = 2y$. Q. E. D.

PROP. V.

THE distance between the Foci is a mean Proportional between the Sum and difference of the Transverse and Conjugate Axe. *i. e.* $AB + DE.KH :: KH.AB - DE$.

DEMONST.

DEMONST.

For KH, put b ; then $KDq - CDq = KCq$,
i. e. $\frac{1}{4}t^2 - \frac{1}{4}c^2 = \frac{1}{4}b^2$, or $t^2 - c^2 = b^2 \therefore t + c$.
 $b :: b. t - c$, or, $AB + DE. KH :: KH. AB -$
 $DE. \mathcal{Q}. E. \mathcal{D}.$

PROP. VI.

A Fourth Proportional to the Conjugate, Transverse, and any Ordinate, is equal to a mean Proportional between the Abscissas of that Ordinate.

DEMONST.

Let the fourth Proportional be b ; then $c. t ::$
 Fig. IV. $y. b \therefore \frac{ty}{c} = b$; But (by Prop. I.) $t^2. c^2 :: t - x$
 $xx. y^2 \therefore$ (by 22. E. 6.) $t. c :: \sqrt{t - x \times x}. y$
 and $\sqrt{t - x \times x} = \left(\frac{ty}{c} =\right) b. \mathcal{Q}. E. \mathcal{D}.$

PROP. VII.

THE distance between the Foci, is a mean Proportional between the Transverse Axe, and the difference of the Transverse Axe, and the Parameter, *i. e.* $AB. KH :: KH. AB - LR.$

DEMON.

DEMONST.

$KDq - CDq = KCq$; that is, $\frac{1}{4}t^2 - \frac{1}{4}c^2 = \frac{1}{4}b^2$, or $t^2 - c^2 = b^2$; But $tp = c^2 \therefore t^2 - tp = b^2$, and $t.b :: b.t - p$, or, $AB.KH :: KH.AB - LR$. Q. E. D.

PROP. VIII.

AS the Square of any Ordinate is to the Rectangle of it's Abfcissas, so is the Square of the Conjugate; to the Square of the Conjugate added to the Square of the distance of the Foci; that is, $FGq. AG \times GB :: EDq. EDq + KHq$.

DEMONST.

$KEq = KCq + CEq$. i. e. $\frac{1}{4}t^2 = \frac{1}{4}b^2 + \frac{1}{4}c^2$, or $t^2 = b^2 + c^2$; but (by Prop. I.) $y^2. t - x \times x :: c^2. (t^2 =) b^2 + c^2$. or, $FGq. AG \times GB :: EDq. EDq + KHq$. Q. E. D.

PROP. IX.

AS the Square of any Ordinate, is to the Rectangle of it's Abfcissa into the Parameter; so is the difference between the Square of the Conjugate Axe, and the Rectangle of the Abfcissa into the Parameter, to the Square of the Conjugate Axe. i. e. $FGq. BG \times LR :: EDq. BG \times LR - EDq$.

D

DEMON.

DEMONST.

Fig. IV. By the Equation of the Curve, $t^2 y^2 = c^2 t x - c^2 x^2$. But $\frac{c^2}{p} = t \therefore$ (by Substitution) $\frac{c^4 y^2}{p^2} = \frac{c^4 x}{p} - c^2 x^2$, and $c^2 y^2 = c^2 p x - p^2 x^2$. *i. e.* $y^2 \cdot p x :: c^2 - p x \cdot c^2$. or, $FGq \cdot BG \times LR :: EDq - BG \times LR$. EDq . \mathcal{Q} . E . \mathcal{D} .

PROP. X.

AS the Square of the Conjugate Axe, is to the Square of the Transverse Axe; so is the Rectangle of any two Abscissas of the Conjugate Axe, to the Square of the Ordinate which dividesthem. *i. e.* $DEq \cdot ABq :: Dh \times hE \cdot Fhq$.

DEMONST.

Let $Eh = x$, and $Fh = y$. then (by Prop. 1.) $ABq \cdot EDq :: AG \times GBq \cdot FGq$. But (by 5. E. 2.) $AG \times GB = \overline{CB}^2 - \overline{Fh}^2$, and $\overline{Ch}^2 = (FGq =) CEq - Dh \times hE$. \therefore (by Substitution.) $ABq \cdot EDq :: CBq - Fhq \cdot CEq - Dh \times hE$, that is, $t^2 \cdot c^2 :: \frac{1}{4}t^2 - y^2 \cdot \frac{1}{4}c^2 - cx + x^2$; which reduced to an Equation, produces $c^2 y^2 = t^2 cx - t^2 x^2$. *i. e.* $c^2 \cdot t^2 :: c - x \times x \cdot y^2$. or $DEq \cdot ABq :: Dh \times hE \cdot Fhq$. \mathcal{Q} . E . \mathcal{D} .

Definition. A third Proportional to the Conjugate, and Transverse Axe, is a Parameter to the Conjugate Axe; that is p , being put for the Parameter c . $t :: t \cdot p \therefore cp = t^2$.

PROP.

PROP. XI.

AS the Conjugate Axe, is to it's Parameter; so is the Rectangle of any two Abscissas of the Conjugate Axe, to the Square of the Ordinate which divides them.

DEMONST.

For t^2 , in the last Equation, put it's equal cp ; then $cy^2 = cp x - p x^2$, i. e. $c.p :: c - x \times x$, y^2 . Q. E. D.

PROP. XII.

AS the Square of any Ordinate of the Conjugate, is to the Rectangle of the Abscissas which it divides; so is the Sum of the Squares of the distance of the Foci, and the Conjugate Axe, to the Square of the Conjugate Axe.

DEMONST.

By the 10. $y^2. cx - x^2 :: t^2. c^2$, and (by 47. Fig. IV. Eu. 1.) $t^2 = b^2 + c^2 \therefore$ (by Substitution) $y^2. cx - x^2 :: b^2 + c^2. c^2$; that is, $Fh q. Dh \times hE :: KHq + EDq. EDq$. Q. E. D.

PROP. XIII.

IN any Tangent to the Ellipse, if, from the point of Contact, an Ordinate be drawn to the Axe, and the Tangent continued to cut the Axe produc'd; then it will hold, as the distance (in the Axe) between the Center and Ordinate, is

is to the Abscissa of that Ordinate; so is the remainder of the Axe, to (the distance between the Ordinate and intersection of the Tangent with the Axe. that is,) the *Subtangent*. viz. CG. GB :: AG. GT.

DEMONST.

Fig. V. Let Fp be an indefinitely small part of the Curve, and continued to cut the Axe produc'd in T; draw the Ordinate FG, and, Parallel to it, pq, draw also Fr Parallel to the Axe, and, for Fr, put n , pr, m , and BT, a . then is Bq = $x + n$, Aq = $t - x - n$, pq = $y + m$, and GT = $a + x$. but (by Similar Δ 's) pr. rF :: FG. GT. i. e. $m.n :: y.x + a \therefore n \times \frac{y}{m} = x + a$, and (by the 2d.) $t.p :: tx - x^2 + tn - 2xn - n^2. y^2 + 2ym + m^2$. Also, $t.p :: tx - x^2. y^2 \therefore ptx - px^2 + ptn - 2pxn = ty^2 + 2tym$ and $ty^2 = ptx - px^2 \therefore ptx - px^2 + ptn - 2xnp - 2tym = (ty^2 =) ptx - px^2$; or, $ptn - 2xnp = 2tym$, and $n = \frac{2tym}{pt - 2px} \therefore x + a = \left(n \times \frac{y}{m} = \frac{2tym}{pt - 2px} \times \frac{y}{m} = \frac{ty^2}{p} \times \frac{2}{t - 2x} = \right.$ (because by the 2d. $tx - x^2 = \frac{y^2 t}{p}$) $\frac{2tx - 2x^2}{t - 2x} = \left. \frac{tx - x^2}{\frac{1}{2}t - x} \therefore \frac{1}{2}t - x.x :: t - x.x + a$; or, CG. GB :: AG. GT. Q. E. D.

PROP.

PROP. XIV.

AS the distance from the Center to the Ordinate drawn from the point of Contact, is to half the Transverse Axe; so is half the Transverse Axe, to the distance from the Center to the concurring of the Tangent with the Axe produc'd. *i. e.* $CG. CB :: CB. CT$.

DEMONST.

$CT = CG + GT$; but $CT = \frac{1}{2}t + a$, $CG = \frac{1}{2}t - x$, and (by the 13) $GT = \frac{tx - x^2}{\frac{1}{2}t - x} \therefore$

$\frac{1}{2}t + a = \left(\frac{1}{2}t - x + \frac{tx - x^2}{\frac{1}{2}t - x} \right) = \frac{\frac{1}{4}t^2}{\frac{1}{2}t - x}$ that is, $\frac{1}{2}t - x. \frac{1}{2}t :: \frac{1}{2}t. \frac{1}{2}t + a$; or, $CG. CB :: CB. CT$. *Q. E. D.*

PROP. XV.

AS the distance from the Center to the Ordinate drawn from the point of Contact, is to half the Transverse; so is the Abscissa of that Ordinate, to the External part of the Transverse; that is, $CG. CB :: GB. BT$.

DEMONST.

By the 14. $\frac{1}{2}t + a = \frac{\frac{1}{4}t^2}{\frac{1}{2}t - x} \therefore \frac{1}{4}t^2 + \frac{1}{2}ta =$

$\frac{1}{2}tx - xa = \frac{1}{4}t^2$, and, $a = \frac{\frac{1}{2}tx}{\frac{1}{2}t - x}$, *i. e.* $\frac{1}{2}t - x. \frac{1}{2}t :: x. a$; or, $CG. CB :: GB. BT$.

PROP.

PROP. XVI.

AS the distance from the Center to the Ordinate drawn from the Point of Contact, is to half the Transverse; so is the greater Abscissa of that Ordinate, to the Transverse Axe, added to the External part; that is, CG. CB :: AG. AT.

DEMONST.

Fig. V. By the 15, $a = \frac{\frac{1}{2}tx}{\frac{1}{2}t - x}$ therefore $t + a =$
 $\left(t + \frac{\frac{1}{2}tx}{\frac{1}{2}t - x}\right) \frac{\frac{1}{2}t^2 - \frac{1}{2}tx}{\frac{1}{2}t - x} \text{ i.e. } \frac{1}{2}t - x. \frac{1}{2}t :: t$
 $- x. t + a. \text{ or, CG. CB :: AG. AT. } \mathcal{Q}. E. \mathcal{D}.$

PROP. XVII.

AS the greater Abscissa of the Ordinate drawn from the point of Contact, is to the Sum of the Transverse and External part; so is the less Abscissa of that Ordinate, to the External part. that is, AG. AT :: GB. BT.

DEMONST.

By the 15, $\frac{1}{2}t - x. \frac{1}{2}t :: x. a.$ and (by 16)
 $\frac{1}{2}t - x. \frac{1}{2}t :: t - x. t + a, \therefore (\text{by Equality})$
 $t - x. t + a :: x. a; \text{ or, AG. AT :: GB. BT.}$

PROP. XVIII.

AS the distance from the Center to the Concurring of the Tangent, is to half the Transverse; so is the External part, to the Abscissa

PART II. Of the ELLIPSE.

31

scissa of the Ordinate from the point of Contact;
that is, $CT. CB :: BT. BG.$

DEMONST.

By the 15. $\frac{1}{2}ta = \frac{1}{2}tx + xa \therefore x = \frac{\frac{1}{2}ta}{\frac{1}{2}t+a}$
and, $\frac{1}{2}t+a. \frac{1}{2}t :: a. x$; or, $CT. CB :: BT. BG.$

PROP. XIX.

AS half the Transverse added to the External part, is to the Transverse added to the External part; so is the External part, to the Subtangent; that is, $CT. AT :: BT. GT.$

DEMONST.

By the 18. $x = \frac{\frac{1}{2}ta}{\frac{1}{2}t+a} \therefore x+a = \left(a + \frac{\frac{1}{2}ta}{\frac{1}{2}t+a}\right)$
 $= \frac{ta+a^2}{\frac{1}{2}t+a}$; and, $\frac{1}{2}t+a. t+a :: a. x+a$. or,
 $CT. AT :: BT. GT. Q. E. D.$

PROP. XX.

AS the greater Abscissa of the Ordinate drawn Fig. V. from the Point of Contact, is to half the Transverse; so is the Subtangent, to the External part. *i. e.* $AG. CB :: GT. BT.$

DEMONST.

By the 15. $\frac{1}{2}t - x = \frac{\frac{1}{2}tx}{a} \therefore t-x = \left(\frac{1}{2}t + \frac{\frac{1}{2}tx}{a}\right) = \frac{\frac{1}{2}ta + \frac{1}{2}tx}{a}$, and, $t-x. \frac{1}{2}t :: x+a. a$.
or, $AG. CB :: GT. BT. Q. E. D. PROP.$

PROP. XXI.

AS the Transverse added to the External part, is to half the Transverse; so is the Subtangent, to the Abscissa. *i. e.* AT. CB :: GT. GB.

DEMONST.

By the 18. $\frac{1}{2}t + a = \frac{\frac{1}{2}ta}{x} \therefore t + a = \left(\frac{1}{2}t + \frac{\frac{1}{2}ta}{x} \right) = \frac{\frac{1}{2}tx + \frac{1}{2}ta}{x}$, and, $t + a. \frac{1}{2}t :: x + a. x$; or AT. CB :: GT. GB. Q. E. D.

PROP. XXII.

THE Ordinate drawn from the Point of Contact, divided by the Subtangent, is equal to the Quotient of the distance between the Center and that Ordinate divided by that Ordinate, Multiplied by the Parameter divided by the Transverse Axe; that is, $\frac{GF}{GT} = \frac{CG}{GF} \times \frac{p}{t}$

DEMONST.

By the 13. $tx - x^2 = \frac{1}{2}t - x \times x + a$; and (by the 2d.) $t.p :: (tx - x^2) \frac{1}{2}t - x \times x + a$. $y^2 \therefore ty^2 = p \times \frac{1}{2}t - x \times x + a$, and if you divide by $x + a$, $\frac{ty^2}{x + a} = p \times \frac{\frac{1}{2}t - x}{x + a}$; and again if you divide by ty , $\frac{y}{x + a} = \left(\frac{p \times \frac{1}{2}t - x}{ty} = \right) \frac{\frac{1}{2}t - x}{y} \times \frac{p}{t}$. Q. E. D.

PROP.

PROP. XXIII.

IF Perpendiculars be drawn from the ends of the Transverse, and from the Center, so as to cut any Tangent, and also if from the Point of Contact, be drawn an Ordinate, these four Lines shall be Proportional; that is, $AO \cdot CP :: GF \cdot BQ$. Fig. VI

DEMONST.

By the 19. $TA \cdot TC :: TG \cdot TB ::$ (by 4. Eu. 6.) $AO \cdot CP :: GF \cdot BQ$. $\mathcal{Q} \quad E. \quad \mathcal{D}.$

COROL.

$$AO \times BQ = CP \times GF.$$

PROP. XXIV.

IF Perpendiculars be drawn from the extremities of the Transverse, and cut any Tangent, then the Rectangle of these Perpendiculars shall be equal to the Rectangle of the greatest and least distance of either of the Foci from the Vertices. *i. e.* $AO \times BQ = AH \times HB = BK \times KA$.

DEMONST.

Let $BQ = m$, $AO = n$, and $AK = HB = q$; then (by Similar Δ 's) $m \cdot y :: (a \cdot x + a ::$
by 20) $\frac{1}{2} t \cdot t - x$; and, $n \cdot y :: (t + a \cdot x + a ::$
by 21) $\frac{1}{2} t \cdot x \therefore m = \frac{\frac{1}{2} t y}{t - x}$ and $n = \frac{\frac{1}{2} t y}{x}$, and
E (by

(by Multiplication,) $mn = \frac{\frac{1}{2}t^2y^2}{t-x \times x} \therefore mn. \frac{1}{2}t^2$
 $:: (y^2. t-x \times x :: \text{by the 2d. } p. t ::) \frac{1}{2}pt. \frac{1}{2}t^2,$
 and $mn = (\frac{1}{2}pt = \text{by 3d.}) t-q \times q. \text{ or, } AO$
 $\times BQ = AH \times HB = BK \times KA. \text{ Q. E. D.}$

LEMMA.

Fig. VII. If, from the ends of the Chord AB, the Perpendiculars AD, BC, be drawn to meet the Circle, then right Lines connecting A and C, B and D, shall be Diameters, and consequently the Point of their Concourse O, will be the Center of the Circle, through which if a right Line be drawn any how, it will make the Alternate Segments of the Perpendiculars equal.

DEMONST.

By Hypothesis the Angles A and B are right
 \therefore (by 31. E. 3.) AC and BD are Diameters,
 and O, the Center; but $\triangle OPD$ is Similar to \triangle
 $OQB \therefore OB. BQ :: OD. DP.$ but $OB = OD$
 $\therefore BQ = PD. \text{ Q. E. D.}$

PROP. XXV.

Fig. VIII. IF, from the Intersections (P, S) of a Circle whose Diameter is the Transverse Axe with any Tangent, Perpendiculars Pk, Sh, be drawn, they shall cut the Transverse Axe in the Focal Points; that is, the Points k, h Coincide with K, H.

DEMON.

DEMONST.

The Δ 's TBQ, ATO are Similar to the Δ 's TSh, TPk, each having a right Angle, and the Angle T, Common \therefore AO. Pk $::$ Sh. BQ and $AO \times BQ = (Pk \times Sh =$ by the precedent Lemma $Pk \times kt$; or, $hr \times Sh =$ by 35. E. 3.) $Ak \times kB$ or $Bh \times hA$. but $AO \times BQ = /K Ak \times KB$ or $AH \times HB$, by the 24 \therefore the Points H, h and K, k are Coincident. Q. E. D.

COROL.

$PK \times SH = \frac{1}{4}pt$; because $PK \times SH = (AK \times KB = t - q \times q =$ by the 3.) $\frac{1}{4}tp$.

PROP. XXVI.

IF to any Point of the Curve right Lines be drawn from the Foci, and one of the Lines be continued; then a right Line Bisecting the External Angle, shall touch the Curve in the Angular point.

DEMONST.

Take $FX = FH$; then (because by the Hy- Fig. IX. pothesis $\angle XFT = \angle TFH$) if you take any Point S, in the Line FT; $HS = XS$, by 4 E. 1. Draw KS, then $KS + (SX =) SH$ is greater than $(KX =) AB$, and \therefore the point S, is without the Curve, for if it were in the Curve $KS + SH$ (by the Genesis $=$) AB .

PROP.

PROP. XXVII.

LINES drawn from the Foci to the Point of Contact, make equal Angles with the Tangent.

DEMONST.

By the 26. $\angle HFT = (\angle XFT = \text{by 15. E. I.})$
 $\angle KFO$. Q. E. D.

PROP. XXVIII.

A Right Line Perpendicular to the Tangent at the point of Contact, Bifects the Angle form'd by Lines drawn from the Foci to the same Point. that is, if FY , be Perpendicular to OT , then, $\angle KFY = \angle HFY$.

DEMONST.

Fig. IX. The $\angle PFY = \angle YFT$ by Hypothesis, from which if you take away the $\angle KFP = \angle HFS$ by the 27. there remains $\angle KFY = \angle YFH$.
 Q. E. D.

PROP. XXIX.

IF, on the Tangent at the Point of Contact, a Perpendicular be drawn, and cut the Axe, it will divide the distance between the Foci, in the same Proportion, as Lines drawn from the Foci to the same Point; *i. e.* $KF : FH :: KY : YH$.

DEMON.

DEMONST.

In the $\triangle KFH$, the $\angle KFY = \angle YFH$ by the 28. \therefore (by the 3d. E. 6.) $KF.FH :: KY.YH$.
Q. E. D.

PROP. XXX.

IF, on the Tangent, at the Point of Contact, a Perpendicular be drawn, and if, from the Point, where that Perpendicular cuts the Axe, Lines be drawn Perpendicular, to Lines drawn from the Foci to the Point of Contact, then the distance on these Lines, from the Point of Contact to the Perpendiculars, will be equal to half the Parameter of the Axe; that is $Fq = Fr = \frac{1}{2}p$.

DEMONST.

From the Points S, P, where a Circle on the Fig. IX. Transverse cuts the Tangent, draw the Lines SH, PK to the Foci, which will be Perpendicular to PT, by 25. and consequently Parallel to FY; continue KF, HS, till they concur in X; then $KX = t$, and $HX = 2HS$ by 26. and because $\triangle KFY$ is Similar $\triangle KXH$, and $\triangle KPF$, Similar $\triangle YFq$. $\therefore KX.XH :: (KF.FY ::) KP.Fq$; and $KX \times Fq = XH \times KP$; and $\frac{1}{2}KH \times Fq = \frac{1}{2}XH \times KP$. that is, $\frac{1}{2}t \times Fq =$ /X
($SH \times KP =$ by 25) $\frac{1}{4}pt$. $\therefore Fq = \frac{1}{2}p$. but
(by 28) $\angle YFq = YFr$, and (by 26. E. 1.) $Fr = Fq$. $\therefore Fq = Fr = \frac{1}{2}p$. Q. E. D.

PROP.

PROP. XXXI.

IF Perpendiculars from the Vertices cut any Tangent, then the part of the Tangent intercepted between the Intersections, shall be the Diameter of a Circle, whose Circumference shall pass through the Foci.

DEMONST.

Fig. X. By the 24. $AO \times BQ = AK \times KB \therefore AO.$
 $AK :: KB. BQ.$ but the Angles OAK , and
 QKB are right, \therefore by 6. E. 6. the Δ 's OAK
 and BQK are Similar, and $\angle AOK = \angle QKB$.
 But $\angle AKO + \angle AOK = L$, $\therefore \angle AKO + \angle QKB$
 $= L$, and consequently (by 13. Eu. 1.) $\angle OKQ$
 $= L$, and (by 31. Eu. 3.) OQ is a Diameter of
 a Circle, whose Periphery will pass through K .
 in like manner the $\angle OHQ$ is prov'd a right
 Angle. *Q. E. D.*

COROL.

If OQ be Bisected in N , then $NO = NQ =$
 $NK = NH$.

PROP. XXXII.

Fig. XI. **I**F, from either Focus, a right Line be drawn
 through the Point of Contact, and continu-
 ed till it be equal to the Transverse Axe, and the
 Extremity connected by a streight Line to the
 other Focus; then the distance between the Cen-
 ter, and the Intersection of the last Line with
 the Tangent, is equal to half the Transverse
 Axe; that is, $CS = CB$.

DEMON.

DEMONST.

In the Δ 's HCS and HKX, the $\angle KHX$ is common, and $KC = CH$, also $HS = SX$ by 26. \therefore (by 6. E. 6.) the Δ 's are Similar and CS is Parallel to KX; also $CS = (\frac{1}{2}XK = \frac{1}{2}AB =) CB$. \mathcal{Q} . E . D .

PROP. XXXIII.

IF, from the Focus, a right Line be drawn to the Point of Contact, and another through the Center Parallel to the Tangent, then the distance between the Point of Contact and the intersection of these Lines is equal to half the Transverse Axe.

DEMONST.

Draw CS Parallel to KF; then is the Figure ZCSF a Parallelogram, and $ZF = (CS = \text{by 32}) BC$. \mathcal{Q} . E . D .

PROP. XXXIV.

IF to the Tangent drawn to the Vertex of any Diameter, a right Line be drawn Parallel, the part of that Line which lies within the Curve shall be Bisected by the Diameter; that is, $xz = bz$, also $\Delta Vxp + \Delta CVo = \Delta BCS = \Delta dpz + \Delta Cdr$. Fig. XII.

DEMONST.

Let $dz = y$, $dp = c$, $Cd = n$, $BS = r$, $dr = p$, $Vx = Y$, $CV = g$, $Vo = q$, $Vp = b$, $Bd =$

$Bd = x$, and $BV = X$, then 1. (from Similar Δ 's) $\frac{y}{c} = \left(\frac{FG}{GT} = \text{by 22.} \right) \frac{CG}{GF} \times \frac{p}{t}$. But

Fig. XII. $\frac{CG}{GF} = \frac{\frac{1}{2}t}{r} \therefore \frac{y}{c} = \frac{\frac{1}{2}t}{r} \times \frac{p}{t}$; divide by $\frac{p}{t}$, and

$\frac{\frac{1}{2}t}{r} = \frac{y}{c} \times \frac{t}{p}$; which Multiplied by cry , gives

$\frac{1}{2}tcy = ry^2 \times \frac{t}{p}$. But $y^2 \times \frac{t}{p} = t - x \times x$ by

thezd. $\therefore \frac{1}{2}tcy = r \times t - x \times x$, and $\frac{1}{2}tcy + rn^2$

$= (r \times t - x \times x + rn^2) = r \times t - x \times x + n^2$;

and (by 5. E. 2.) $t - x \times x + n^2 = \frac{1}{4}t^2 \therefore \frac{1}{2}tcy$

$+ rn^2 = r \times \frac{1}{4}t^2$. Divide by $\frac{1}{2}t$, and $cy + \frac{rn^2}{\frac{1}{2}t}$

$= r \times \frac{1}{2}t$; by Similar Δ 's $\frac{1}{2}t.r :: n.p \therefore \frac{rn}{\frac{1}{2}t} =$

$p \therefore$ (by Substitution) $cy + pn = r \times \frac{1}{2}t$, or dp

$\times dz + dr \times Cd = BS \times BC$; that is, ΔBSC

$= \Delta dpz + Cdr$.

2. By Similar Δ 's $\frac{Y}{b} = \left(\frac{FG}{GT} = \text{by 22} \right)$

$\frac{CG}{GF} \times \frac{p}{t}$; but $\frac{CG}{GF} = \frac{g}{q} \therefore \frac{Y}{b} = \frac{g}{q} \times$

$\frac{p}{t}$ divide by $\frac{p}{t}$, and $\frac{g}{q} = \frac{Y}{b} \times \frac{t}{p}$, and by

Multipling by bqY ; $gbY = qY^2 \times \frac{t}{p}$.

But $tX - X^2 = Y^2 \times \frac{t}{p} \therefore gbY = q \times tX - X^2$;

and $gbY + qg^2 = q \times tX - X^2 + g^2$. but (by

5. E. 2.) $tX - X^2 + g^2 = \frac{1}{4}t^2 \therefore gbY + g^2q =$

$q \times \frac{1}{4}t^2$; divide by g , and $bY + qg = \frac{q \times \frac{1}{4}t^2}{g}$ but

$g.q :: \frac{1}{2}t.r :: \frac{\frac{1}{2}tq}{g} = r.$ and (by Substitution)

$bY + qg = r \times \frac{1}{2}t$, that is, $Vp \times Vx + Vo \times CV = CB \times BS$, or $\triangle Vxp + \triangle CVo = (\triangle BCS = \text{by the 1st part}) \triangle dpz + \triangle Cdr$.

3. From both sides of the last Equation take $\triangle pbC$, and there remains $\triangle b zr =$ and Similar to $\triangle obx :: xb = bz$. Q. E. D.

PROP. XXXV.

THE same things being suppos'd as in the last, the $\triangle BSC = \triangle CFT$; also the *Trapezium* $dBSr = \triangle pdz$; the *Trapezium* $FbpT = \triangle b zr$; and $FT + bp \times bF = zb \times br$.

DEMONST.

From Similar \triangle 's $BS.FG :: (BC.GC :: \text{by Fig. XII. 14.}) CT.BC :: BS \times BC = FG \times CT$, or $\triangle BSC = \triangle CFT = (\text{by Prop. 34.}) \triangle pdz + \triangle Cdr$; From the 1st. and 3d. Equ. take $\triangle Cdr$, and there remains the *Trapezium* $dBSr = \triangle pdz$; Also, from the 2d. and 3d. take the $\triangle pbC$, and there remains the *Trapezium* $FbpT = \triangle b zr :: (\text{by the Lemmato Prop. 11. of the Parabola}) FT + pb \times bF = zb \times br$. Q. E. D.

LEMMA.

The same things being still supposed, $Yb \times nT = zb \times br$; For $YC = (FC). bC :: FT.bp$, and by Composition $YC + bC. bC :: FT + bp.bp$. and (by Alternation $YC + bC =$) $Yb.FT + bp :: (bC.bp ::) np = bF$. $nT :: Yb$
F x

$\times nT = \overline{FT + bp} \times bF =$ by Lemma to Prop. II. Part I.) $zb \times br$.

Definition. If $(FS. FQ ::) br. bz :: 2FT$. P , the Parameter belonging to the Diameter FY , then $P = bz \times \frac{2FT}{br}$, and,

PROP. XXXVI.

AS any Diameter is to it's Parameter (so obtained) so is the Rectangle of any Abscissas of that Diameter, to the Square of the Ordinate which divides them. that is, (if you put D , for the Diameter FY , x for the Abscissa Fb , and y for the Ordinate $bz = bx$. then) $D.P :: D - x \times x. y^2$.

DEMONST.

By the *Definition*, $P = y \times \frac{2FT}{br}$ therefore

$$\frac{P \times \overline{D - x} \times x}{D} = \left(\frac{y \times 2FT}{br} \times \frac{\overline{D - x} \times x}{D} = \right)$$

 $\frac{y}{br} \times \overline{D - x} \times x \times \frac{2FT}{D}$: But $\frac{2FT}{D} = \left(\frac{FT}{\frac{1}{2}D} = \right)$
 by Similar Δ 's) $\frac{Tn}{(np =) x}$, and, (by Substitution)

$$P \times \frac{\overline{D - x} \times x}{D} = \left(\frac{y}{br} \times \overline{D - x} \times Tn = \right)$$
 (by the preceding Lemma) $\frac{y}{br} \times y \times br = y^2$, and \therefore
 $D.P :: \overline{D - x} \times x. y^2$. *Q. E. D.*

PROP.

PROP. XXXVII.

AS any Parameter is to it's Diameter, so is the Square of it's Conjugate, to the Square of the Diameter. *i. e.* $P. YF :: FYq. XDq.$

DEMONST.

In this Case $x = \frac{1}{2}D$, and $y = \frac{1}{2}C$ \therefore by the last D. $P :: (\frac{1}{4}D^2. \frac{1}{4}C^2 ::) D^2. C^2$, or $P. YF :: FYq. XDq.$

COROL.

Hence any Conjugate Diameter is a mean Proportional between the Diameter to which it is Conjugate, and the Parameter of that Diameter, for by this Prop. $DP = C^2 \therefore D. C :: C. P.$

PROP. XXXVIII.

IF a Tangent cut any Diameter, and if, from the Point of Contact, an Ordinate be drawn to that Diameter, then as the distance between that Ordinate and Center is to the Abscissa; so is the Diameter less by the Abscissa, to the Subtangent on the Diameter continued. that is, $CP. PF :: YP. PT.$ Fig XIII.

DEMONST.

Let GQ , be an indefinitely small part of the Curve, and continued till it cut the Diameter produced in T . Draw the Ordinate GP , and Parallel to it Qo , and Gr , Parallel to the Diameter
eter

eter FY. Put $YF = D$, $FP = x$, $GP = y$, $Gr = n$, $Qr = m$, and $FT = a$. then $YP = D - x$, $Yo = D - x - n$, $oF = x + n$, $Qo = y + m$, and $PT = a + x$. then (by Similar Δ 's) $m.n :: y.a + x \therefore a + x = n \times \frac{y}{m}$ and (by the 36.) $D.P. Dx - x^2.y^2$. also $D.P :: Dx - x^2 + Dn - 2xn - n^2.y^2 + 2ym + m^2 \therefore Dy^2 = PDx - Px^2$, and $Dy^2 + 2Dym = PDx - Px^2 + PDn - 2Pxn \therefore PDx - Px^2 + PDn - 2Pxn - 2Dym = (Dy^2 =) PDx - Px^2$, and $PDn - 2Pxn = 2Dym$, therefore $n = \frac{2Dym}{PD - 2Px}$. But $a + x = n \times \frac{y}{m} \therefore a + x = (\frac{2Dym}{PD - 2Px} \times \frac{y}{m} = \frac{Dy^2}{P} \times \frac{2}{D - 2x} =$ (because by 36. $\frac{Dy^2}{P} = Dx - x^2$.) $Dx - x^2 \times \frac{2}{D - 2x} = \frac{2Dx - 2x^2}{D - 2x} =) \frac{Dx - x^2}{\frac{1}{2}D - x}$; that is, $\frac{1}{2}D - x.x :: D - x.x + a$, or, CP. PF :: YP. PT. Q. E. D.

PROP. XXXIX.

Fig. XIII.

IF a Tangent intersect any Diameter, and, from the Point of Contact, an Ordinate be drawn to that Diameter; As the Semi-Diameter less by the Abscissa, is to the Semi-Diameter; So is the Semi-Diameter, to the Semi-Diameter added to the External part of the Diameter produced to the Intersection of the Tangent. i. e. CP. CF :: CF. CT.

DEMON.

DEMONST.

$CP + PT = CT$. But $CP = \frac{1}{2}D - x$, and
 PT , (by the last is) $= \frac{Dx - x^2}{\frac{1}{2}D - x}$; also $CT =$
 $\frac{1}{2}D + a$. $\therefore \frac{1}{2}D + a = \left(\frac{1}{2}D - x + \frac{Dx - x^2}{\frac{1}{2}D - x} \right)$
 $\frac{\frac{1}{4}D^2}{\frac{1}{2}D - x}$, and $\frac{1}{2}D - x$. $\frac{1}{2}D :: \frac{1}{2}D$. $\frac{1}{2}D + a$; or,
 CP . $CF :: CF$. CT . \mathcal{Q} . E . D .

PROP. XL.

THE same things being suppos'd as in the
 last, it will be, as the Semi-Diameter less
 by the Abscissa, is to the Semi-Diameter; so is
 the Abscissa, to the External part of the Diamo-
 ter produc'd to the intersection of the Tangent.
i. e. CP . $CF :: PF$. FT .

DEMONST.

By the 39. $\frac{\frac{1}{4}D^2}{\frac{1}{2}D - x} = \frac{1}{2}D + a \therefore \frac{1}{4}D^2 = \frac{1}{2}D^2$
 $+ \frac{1}{2}Da - \frac{1}{2}Dx - xa$, and $a = \frac{\frac{1}{2}Dx}{\frac{1}{2}D - x}$; that
 is, $\frac{1}{2}D - x$. $\frac{1}{2}D :: x$. a . or, CP . $CF :: PF$. FT .

PROP. XLI.

AS the Semi-Diameter less by the Abscissa is
 to the Semi-Diameter, so is the Diameter
 less by the Abscissa; to the Diameter added to
 the External part of the Diameter produced to
 the Tangent. *i. e.* CP . $CF :: YP$. YT .

DEMON.

DEMONST.

By the 40, $a = \frac{\frac{1}{2}Dx}{\frac{1}{2}D - x}$, $\therefore D + a = (D + \frac{\frac{1}{2}Dx}{\frac{1}{2}D - x}) \frac{\frac{1}{2}D - x}{\frac{1}{2}D - x} = \frac{\frac{1}{2}D^2 - \frac{1}{2}Dx}{\frac{1}{2}D - x}$; *i. e.* $\frac{1}{2}D - x. \frac{1}{2}D :: D - x. D + a$, or, CP.CF :: YP. YF. Q. E. D.

/ T

PROP. XLII.

Fig.
XIII.

AS the Diameter less by the Abscissa, is to the Diameter added to the External part, so is the Abscissa, to the External part of the Diameter produc'd to the Tangent; that is, YP. YT :: PF. FT.

DEMONST.

By the 40, $\frac{1}{2}D - x. \frac{1}{2}D :: x. a$; and (by the 41.) $\frac{1}{2}D - x. \frac{1}{2}D :: D - x. D + a$ \therefore (by Equality) $D - x. D + a :: x. a$. or, YP. YT :: PF. FT.

PROP. XLIII.

AS the Semi-Diameter added to the External part, is to the Semi-Diameter; so is the External part, to the Abscissa; *i. e.* CT. CF :: FT. FP.

DEMONST.

By the 40, $\frac{1}{2}Da - xa = \frac{1}{2}Dx \therefore x = \frac{\frac{1}{2}Da}{\frac{1}{2}D + a}$
that is, $\frac{1}{2}D + a. \frac{1}{2}D :: a. x$. or, CT. CF :: TF. PF.

PROP.

PROP. XLIV.

AS the Semi-Diameter added to the External part, is to the Diameter added to the external part; so is the External part, to the Subtangent. *i. e.* CT. YT :: FT. PT.

DEMONST.

By the 43. $x = \frac{\frac{1}{2}D a}{\frac{1}{2}D + a}$ therefore $x + a =$
 $(\frac{\frac{1}{2}D a}{\frac{1}{2}D + a} + a) = \frac{D a + a^2}{\frac{1}{2}D + a}$; that is, $\frac{1}{2}D + a. D + a$
 $:: a. x + a$, or, CT. YT :: FT. PT. *Q. E. D.*

PROP. XLV.

AS the Semi-Diameter added to the External part, is to the Semi-Diameter; so is the Diameter added to the External part, to the Diameter less by the Abscissa. that is, CT. CF :: YT. YP.

DEMONST.

By the 41. $\frac{1}{2}D - x. \frac{1}{2}D :: D - x. D + a$, and
 (by the 39.) $\frac{1}{2}D - x. \frac{1}{2}D :: \frac{1}{2}D. \frac{1}{2}D + a$, $\therefore \frac{1}{2}D$
 $+ a. \frac{1}{2}D :: D + a. D - x$; or, CT. CF :: YT.
 YP. *Q. E. D.*

PROP. XLVI.

AS the Diameter less by the Abscissa, is to the Semi-Diameter, so is the Subtangent; to the External part of the Diameter produced to the Tangent; that is, YP. CF :: PT. FT.

DEMONST.

Fig.
XIII.

DEMONST.

By the 40. $\frac{1}{2}Da - xa = \frac{1}{2}Dx$, $\therefore \frac{1}{2}D - x = \frac{\frac{1}{2}Dx}{a}$; and $D - x = \left(\frac{1}{2}D + \frac{\frac{1}{2}Dx}{a} \right) = \frac{\frac{1}{2}Da + \frac{1}{2}Dx}{a}$, that is, $D - x. \frac{1}{2}D :: a + x.a$; or, YP. CF :: PT. FT. Q. E. D.

PROP. XLVII.

AS the Diameter added to the External part, is to the Semi-Diameter; so is the Subtangent, to the Abscissa. *i. e.* YT. CF :: PT. PF.

DEMONST.

By the 40. $\frac{1}{2}Da = \frac{1}{2}Dx + xa$, therefore $\frac{1}{2}D + a = \frac{\frac{1}{2}Da}{x}$; and $D + a = \left(\frac{1}{2}D + \frac{\frac{1}{2}Da}{x} \right) = \frac{\frac{1}{2}Dx + \frac{1}{2}Da}{x}$; that is, $D + a. \frac{1}{2}D :: x + a.x$; or YT. CF :: PT. PF. Q. E. D.

PROP. XLVIII.

IF, from the Extremities of two Conjugate Diameters, Ordinates be drawn to the Axe; then the distance on the Axe between the Center and one of these Ordinates, is a mean proportional between the Segments of the Axe made by the other Ordinate; that is, AG. CH :: CH. GB; or, AH. CG :: CG. HB.

DEMONST.

DEMONST.

Draw the Tangent FT, which will be Parallel to CD by the 34. And let $BC = t$, $CH = a$, $GT = s$, and $CG = x$, then $GB = t - x$, and $AG = t + x$; and (by the 4. and 22. E. 6.) $GTq. CHq :: (FGq. DHq :: \text{by 1.}) AG \times GB. AH \times HB$. But (by 5. E. 2.) $AG \times GB = BCq - CGq$, and $AH \times HB = BCq - CHq$; $\therefore GTq. CHq :: CBq - CGq. CBq - CHq$; that is, $s^2. a^2 :: t^2 - x^2. t^2 - a^2$. But (by the 13.) $CG. GB :: AG. GT$; that is, $x.$

Fig.
XIV.

$t - x :: t + x. s$. therefore $s = \frac{t^2 - x^2}{x}$, and $s^2 = \frac{t^2 - x^2 \times t^2 - x^2}{x^2}$, consequently $\frac{t^2 - x^2 \times t^2 - x^2}{x^2}$

$t^2 - x^2 :: a^2. t^2 - a^2$, and if you divide the two first Terms by $t^2 - x^2$; $\frac{t^2 - x^2}{x^2}. 1 :: a^2. t^2 - a^2$

and by Composition, $t^2. a^2 :: \left(1 + \frac{t^2 - x^2}{x^2} =\right)$

$\frac{t^2}{x^2}. \frac{t^2 - x^2}{x^2}$, and if you Multiply the two last

Terms by x^2 , $t^2. a^2 :: t^2. t^2 - x^2 \therefore a^2 = t^2 - x^2$ and $t + x. a :: a. t - x$; or, $AG. CH :: CH. GB$. In like manner we may prove that $AH. CG :: CG. HB$.

COROL. I.

Hence it is easie to draw a Conjugate Diameter, without drawing a Tangent. For, if you produce the Ordinate FG, to I in the Circumference of a Circle on the Transverse Axe, and make $CH = GI$, then from H, draw the Or-

G

dinate

dinate HD, and lastly from the point D, through the Center draw DCX, and it will be the Conjugate Diameter required.

COROL. II.

The Sum of the Squares of any two Diameters, as (DX and FY) is equal to the Sum of the Squares of the Transverse and Conjugate Axes.

For if a . be put for CG, then (by Prop. I.) $t^2 \cdot c^2 :: (AH \times HB = \text{by this Prop. } CGq =) a^2 \cdot HDq = \frac{c^2 a^2}{t^2}$, and (by this Prop.) $CHq = (AG \times GB =) \frac{1}{4} t^2 - a^2$; \therefore (by 47. E. I.) $CDq = \frac{1}{4} t^2 - a^2 + \frac{c^2 a^2}{t^2}$; also (by Prop. I.) $t^2 \cdot c^2 :: (AG \times GB =) \frac{1}{4} t^2 - a^2 \cdot GFq = \frac{1}{4} c^2 - \frac{a^2 c^2}{t^2}$ $\therefore CFq = a^2 + \frac{1}{4} c^2 - \frac{a^2 c^2}{t^2}$ whence, $CDq + CFq = \frac{1}{4} t^2 + \frac{1}{4} c^2$.

PROP. XLIX.

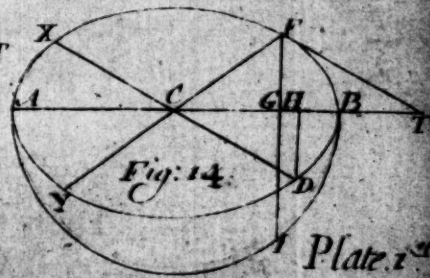
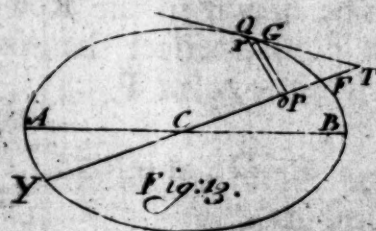
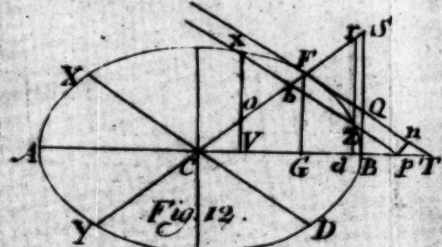
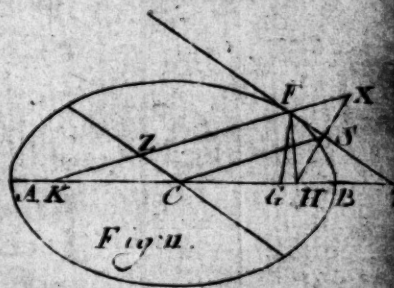
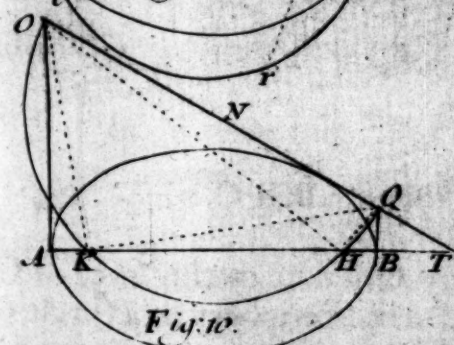
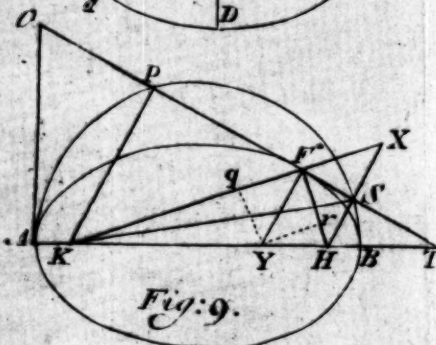
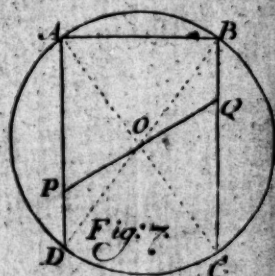
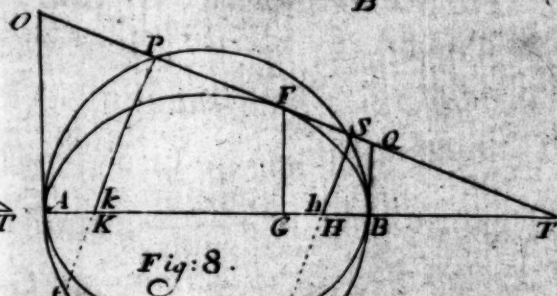
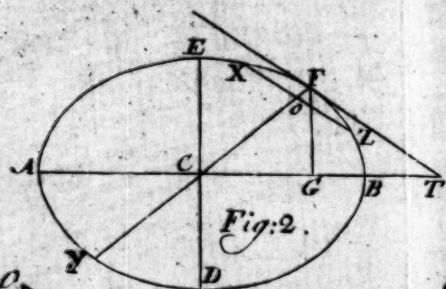
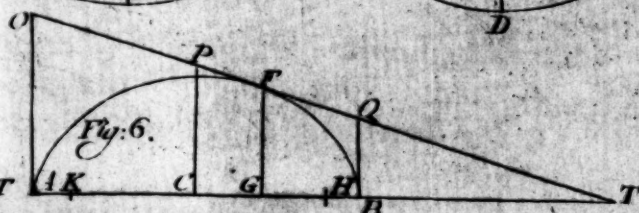
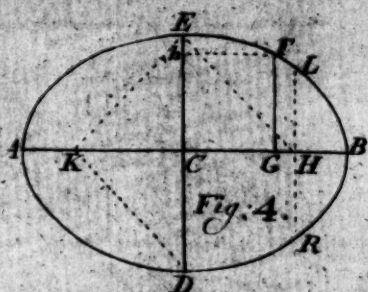
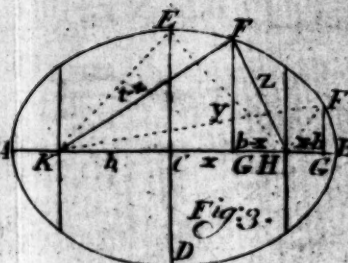
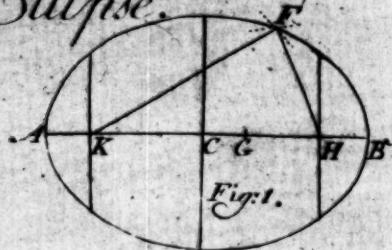
Fig.
XV.

IF any Ordinate to the Axe, as GF be produced to the Periphery of a Circle on the Transverse Axe as to I, and if from the points F and I, Tangents be drawn to the Respective Curves, they will both intersect the Axe produced in one and the same point T.

DEMONST.

Draw the Radius CI, and put $BG = x$, $AB = t$, $BT = a$, and $GI = y$. then, $TG \times GC = (GIq =) GB \times GA$; that is, $a + x \times \frac{1}{2} t - x =$

Ellipse.



$= (y^2 =) x \times t - x \therefore \frac{1}{2}tx = \frac{1}{2}ta - ax$, or
 $\frac{1}{2}t - x. \frac{1}{2}t :: x.a$. But in (the Ellipse, by the 15.)
 $\frac{1}{2}t - x. \frac{1}{2}t :: x.a$. In both Curves the three first
 Terms are the same, therefore the fourth Term,
viz. a , = BT, is the same; and consequently
 the point T, is that wherein both Tangents
 will intersect. Q. E. D.

COROL. I.

Hence any point in the Curve being given,
 we have an easie Method of drawing a Tangent
 to touch that point; for, if, from the given
 point F, you draw the Ordinate FG, and pro-
 duce it to the Periphery of the Circumscribing
 Circle in the point I, and draw a Tangent touch-
 ing the Circle in that point as IT, then the point
 T, where that Tangent cuts the Axe produced,
 is the point, to which if, from the given point
 in the Ellipse (*viz.* F) you draw a right Line,
 it will be a Tangent.

COROL. II.

Hence also if, from a point T, given in the
 Axe produced, it be required to draw a Tangent
 to the Ellipse, 'tis easily done. For if, on CT,
 you describe the Semi-Circle CIT, and observe
 its intersection I, with the Circle described on
 the Transverse Axe; then if BZ be made equal
 to BI, and IZ, be drawn, and if, from the point
 F, where that Line cuts the Curve, the streight
 Line TF be drawn, it will touch the Curve in
 the point F.

Scholium. From this Proposition it is evident,
 that all the Properties of Tangents which have
 been

been Demonstrated in the Ellipse from Prop. 13. to Prop. 21. inclusively, hold good also in the Circle.

PROP. L. *Problem.*

Fig.
XVI.

FROM any given point as T, any where without the Ellipse to draw a Tangent,

Construction. From the given point T, through the Center draw the right Line TFCY; and, to the Diameter YF (by Coroll. to Prop. 48) draw the Conjugate Diameter DX, then at pleasure make the Angle YTS, and on TS, set $TR = TC$ and $SR = CF$; join RF, and, Parallel to it, draw SP; Lastly, through P, and, Parallel to the Conjugate Diameter, draw GN; then, if, from the point T, to G or N right Lines be drawn, they will touch the Ellipse in those points.

DEMONST.

By *Construction* and 2. Eu. 6. $TR : RS :: TF : FP$. But $TR = TC$ and $RS = CF$ $\therefore TC : CF :: FT : FP$, and (by Prop. 43.) TG, or TN are Tangents.

PROP. LI.

Fig.
XVII.

IF any Ordinate to the Axe (as Vx) be continued to a point (N,) in the Focal Tangent (TO) then the distance (VN) from the Axe to that point in the Tangent, shall be equal to (Kx) the distance from the Focus to the extremity of that Ordinate.

DEMON.

DEMONST.

Put $CK = b$, $BC = c$, $CV = d$, then $AK = b + c$, $BK = c - b$, $VK = b \pm d$, $BV = c \pm d$, and $AV = c \mp d$, and K being the Focus, (by the 4.) KL will be half the Parameter of the Axe: and (by the 3d.) $CB. AK :: KB.KL$ or $c. c + b :: c - b. \frac{c^2 - b^2}{c} = KL = \frac{1}{2}p$; Also

Fig.
XVII,
/d

$CK. CB :: CB. CT$, or $b. c :: c. \frac{c^2}{b} = CT$,

by the 14. But $CT - CK = KT$; that is, $\frac{c^2}{b} - b = \frac{c^2 - b^2}{b} = KT$, Also $CT \pm CV = VT$,

that is, $\frac{c^2}{b} \pm d = \frac{c^2 \pm b d}{b} = VT$. But (by Si-

milar Δ 's) $KT. KL :: VT. VN$, or, $\frac{c^2 - b^2}{b} :: \frac{c^2 \pm b d}{b} \cdot \frac{c^2 \pm b d}{c} = VN$.

2. By the 2d. $CB. KL :: AV \times VB. Vxq$;

or, $c. \frac{c^2 - b^2}{c} :: c^2 - d^2. \frac{c^4 - b^2 c^2 - c^2 d^2 + d^2 b^2}{c^2}$

$= Vxq$; and $VKq = b^2 \pm 2bd + d^2$. But VKq

$+ Vxq = Kxq$. that is, $\frac{c^4 \pm 2bdc^2 + d^2 b^2}{c^2}$

$= Kxq$, and (by extracting the Square Root,)

$\frac{c^2 \pm bd}{c} = Kx =$ (by the first Part) VN .

Q. E. D.

COR.

COROL.

The Conjugate Axe continued from the Center to the Focal Tangent, is equal to the Semi-Transverse Axe. *i. e.* $CZ = (KE =) CB$.

PROP. LII.

IF Perpendiculars be drawn from the Vertices to the Focal Tangent, then these Perpendiculars shall be equal to the distance, (in the Axe) from each Vertex to it's adjacent Focus respectively; that is. $AO = AK$, and $BQ = BK$.

DEMONST.

By the 24. $AO \times BQ = AK \times KB \therefore AO \cdot KA :: BK \cdot BQ$. But $AO = AK$. by 51. $\therefore KB = BQ$. *Q. E. D.*

PROP. LIII.

IF, from the point of Contact of the Focal Tangent, a right Line be drawn to the Vertex, and any Ordinate be produced to the Tangent and cut that Line, then the distance between the Tangent and intersection of these Lines, is equal to the distance (in the Axe) from the Focus to the Application of the Ordinate. *i. e.* $DN = KV$.

DEMONST.

The $\triangle LDN$ is Similar to $\triangle LOA \therefore OA \cdot DN :: (LO \cdot LN ::) KA \cdot KV$. But by 51. $AO = AK \therefore DN = KV$. *Q. E. D.* PROP.

PROP. LIV.

IF, from any point (P,) of the Conjugate Axe a right Line PO, equal to the difference of the Semi-Transverse and Semi-Conjugate, be applied to the Transverse Axe, and from thence continued, so that the External part OF, be equal to the Semi-Conjugate Axe, then, I say, the Extremity F, of that Line shall be in the Curve of the Ellipse. Fig. XVIII.

DEMONST.

Put $CO = b$, $OG = d$, $CG = (b + d) = x$, and the other Symbols as usual; then $PO = \frac{1}{2}t - \frac{1}{2}c$, and $OF = \frac{1}{2}c$. Then (from Similar Δ 's,) $b : d :: \frac{1}{2}t - \frac{1}{2}c : \frac{1}{2}c$, and (by Composition) $(b + d) : x : d :: \frac{1}{2}t : \frac{1}{2}c \therefore x^2 : d^2 :: \frac{1}{4}t^2 : \frac{1}{4}c^2$, and $\frac{\frac{1}{4}c^2 \times x^2}{\frac{1}{4}t^2} = (d^2 = \text{by 47. E. 1.}) \frac{1}{4}c^2 - y^2$, conse-

quently $y^2 = \left(\frac{1}{4}c^2 - \frac{\frac{1}{4}c^2 \times x^2}{\frac{1}{4}t^2} \right) = \frac{\frac{1}{4}t^2 - x^2 \times \frac{1}{4}c^2}{\frac{1}{4}t^2}$,
 $\therefore \frac{1}{4}t^2 : \frac{1}{4}c^2 :: \frac{1}{2}t + x \times \frac{1}{2}t - x : y^2$. or ACq. EDq
 $\therefore AG \times GB. GFq. Q. E. D.$

PROP. LV.

IF a Circle be drawn on the Transverse Axe of the Ellipse, and Ordinates be drawn to both Curves; it will be, as the Transverse Axe is to the Conjugate, so is any Ordinate in the Circle, to it's corresponding Ordinate in the Ellipse. that is, AB. DE :: sq. sr. Fig. XIX.

DEMONST.

DEMONST.

By 1. $ABq. DEq :: (As \times sB = \text{by 35. E.}$
 3.) $sq^2. sr^2 :: AB. DE :: sq. sr. Q. E. D.$

PROP. LVI.

Fig.
XIX.

AS the Transverse Axe is to the Conjugate Axe; so is the Area of a Circle on the Transverse Axe, to the Area of the Ellipse.

DEMONST.

In the following Demonstrations let $\odot t$, be the Circle on the Transverse, $\odot c$, the Circle on the Conjugate, \bigcirc the Ellipse, and $\odot \sqrt{tc}$ the Circle whose Diameter is the Square Root of t into c , then,

By the 55. $t. c :: (sq. sr :: \text{by 12. E. 5. all the } sq's. \text{ all the } sr's) \odot t. \bigcirc. Q. E. D.$

PROP. LVII.

THE Area of every Ellipse is equal to the Area of a Circle, whose Diameter is the Square Root of the Transverse Axe into the Conjugate.

DEMONST.

By the 56. $\odot t. \bigcirc :: (t. c :: t^2. ct :: \text{by 2 E. 12.}) \odot t. \odot \sqrt{tc} :: \bigcirc = \odot \sqrt{tc}. Q. E. D.$

COR.

COROL.

$\odot t. t^2 :: \odot. tc$; that is, as Circles are to the Squares of their Diameters, so are Ellipses to the Rectangles under their Transverse and Conjugate Axes.

PROP. LVIII.

EVERY Ellipse is a mean Proportional between the Circle on it's Transverse and the Circle on it's Conjugate Axe.

DEMONST.

By 56. $\odot t. \odot :: (t.c :: tc.c^2 :: \text{by 2. E. 12.}$
 $\odot \sqrt{tc}. \odot c :: \text{by 57.}) \odot. \odot c. \mathcal{Q}. E. D.$

PROP. LIX.

ELLIPSES are to each other in a Ratio ^{Fig. XX, & XXI.} compounded of the Subduplicate Ratio of their Parameters, and Sefquiplicate Ratio of their Transverse Axes directly.

DEMONST.

By the 57th. $E = \odot \sqrt{TC}$, and $e = \odot \sqrt{tc} \therefore$
 $E.e :: (\odot \sqrt{TC}. \odot \sqrt{tc} :: \text{by 2. E. 12. } TC. tc$
 $:: \text{because } c = \sqrt{tp}) T^{\frac{3}{2}} \times P^{\frac{1}{2}}. t^{\frac{3}{2}} \times p^{\frac{1}{2}}. \mathcal{Q}. E. D.$

PROP. LX.

Parallelograms drawn with their sides Parallel to the Conjugate Diameters, and Circumscribing the Ellipse are equal. ^{Fig. XXII.}

H

DEMON.

DEMONST.

On the Transverse Axe describe the Circle kND ; continue the Ordinate through the point of Contact to I ; draw the Ordinate MX , and Cd Perpendicular to the Tangent; then, I say, $Cd \times CM = Cx \times Ck$. For, if, for $GI =$ (by 48.) CX we put b , $Cn = d$, $CM = D$, $Cd = p$, $GF = y$, $Cx = c$, and $Ck = t$. then,

By Prop. the 55 th. $y. b :: c. t :: y = \frac{bc}{t}$,
and (by the 39) $\frac{bc}{t} (=y). c :: c. d$. also by Similar Δ 's $b. D :: p. d :: \frac{ct}{b} = (d =) \frac{Dp}{b}$, or, $ct = Dp$. i. e. $Cd \times CM = Cx \times Ck$. $\mathcal{Q}. E. D.$

PROP. LXI.

Fig.
XVII.

AS the distance between the Foci, is to the Transverse Axe, so is the distance between the Focus and the Vertex; to the distance between the Vertex and intersection of the Focal Tangent with the Axe produced. i. e. $KH. AB :: BK. BT$.

DEMONST.

By the 17. $AK. BK :: AT. BT :: AK = (BK =) AH. BK :: AT = BT. BT$; that is, $HK. BK :: AB. BT$. $\mathcal{Q}. E. D.$

PROP.

PROP. LXII.

IF a right Line be drawn from the Focus to any point of the Curve, and, from that point, a Line be drawn \perp to the Axe, and continued to the Perpendicular which cuts the Axe produced in the point of Intersection of the Focal Tangent; then these two Lines are in a constant Ratio, *viz.* as the distance between the Foci is to the Transverse Axe. *i. e.* $KE. En :: KH. AB$. Fig. XVII.

DEMONST.

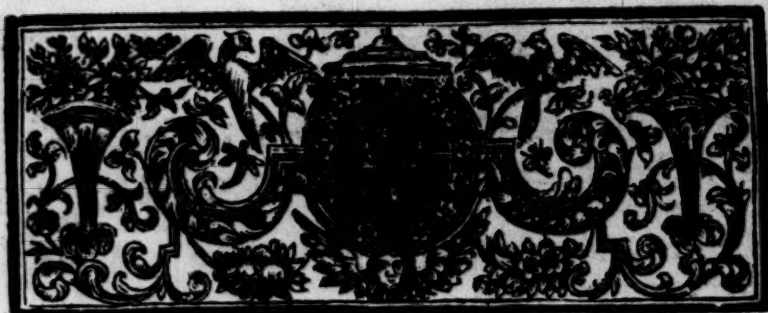
By Prop. 51. $Cz = KE$ and by 52. $BQ = BK$; but (from Similar Δ 's) $Cz. CT :: BQ. BT$; that is, $KE. En :: (BK. BT ::$ by the 61.) $HK. AB$. $\mathcal{Q}. E. D.$

PROP. LXIII.

THE Focal distance of any point in the Curve, is to a Perpendicular let fall from that Focus to the Tangent of the said point, as the Semi-Conjugate Diameter; to the Semi-Conjugate Axis.

DEMONST.

The Triangles FHI, FKL are Similar $\therefore HF. FK :: HI. LK$, that is, $HF + FK. HI + LK :: BC. CO :: FH. HI :: BC \times CD. CO \times CD$; but (by Prop. 60.) $OC \times CD = BC \times CE$, $\therefore FH. HI :: BCD. BCE :: CD. CE$. $\mathcal{Q}. E. D.$ Fig. XXII.



Conic Sections.

PART III.

Of the HYPERBOLA.

The GENESIS.

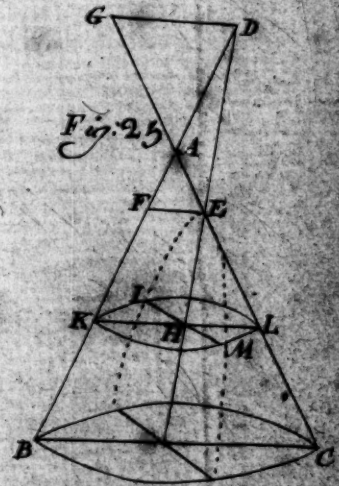
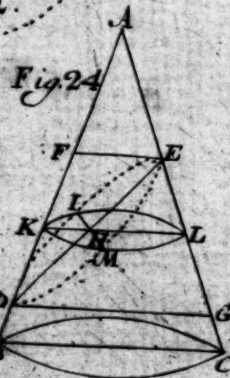
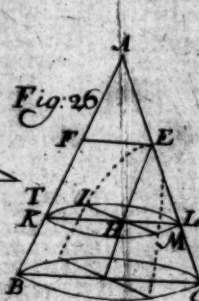
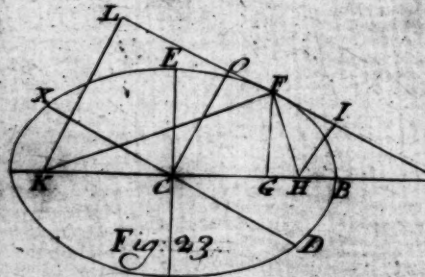
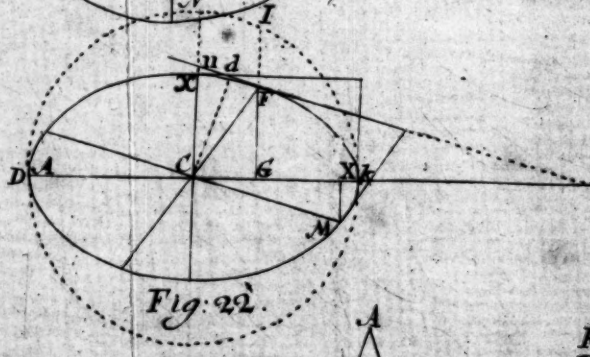
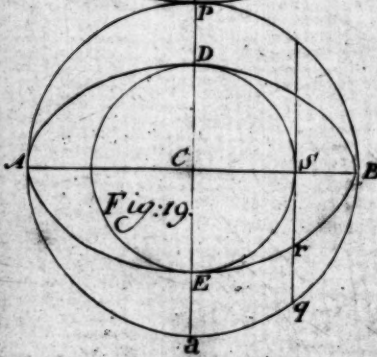
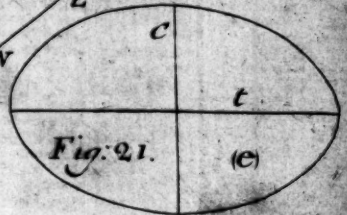
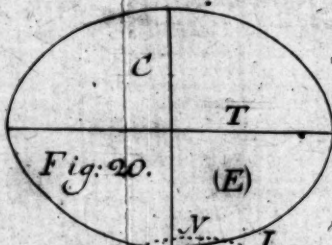
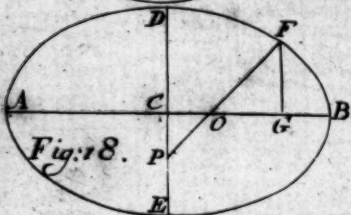
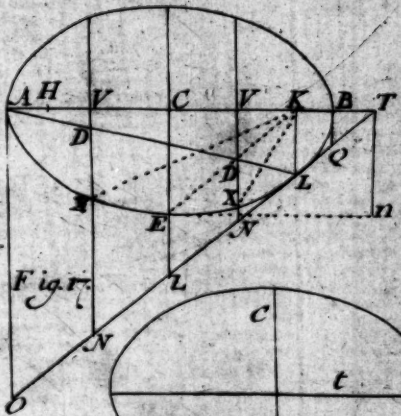
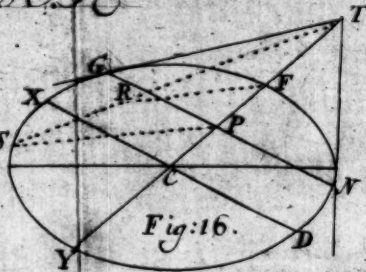
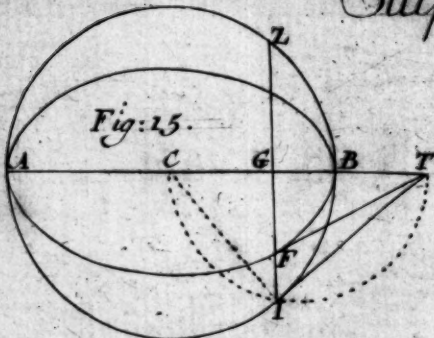
Fig. I.



F, upon a Plane any streight Line AB be taken, and, in that Line continued both ways BK be $= AH$, and the point G , taken any where in that Line (without H and K) and then if with the Radius AG from the point H , as a Center, you describe an Arc, and, with the Radius BG , from the Center K , you intersect the former Arc at F ; and then if from the points H , and K , you draw the Lines HF and FK , I say $HF = FK = AB$.

For

Ellipse



PART III. Of the HYPERBOLA. 61

For, by Construction, $HF = (AG =) AB + BG$; and $FK = BG \therefore HF - FK = (AB + BG - BG =) AB$. in like manner, an indefinite Number of Points may be found; and the Curve Line drawn through them is called an **HYPERBOLA**.

DEFINITIONS.

1. The points H, and K, are called the Focus Fig. I. points.
2. A Diameter of the Hyperbola is a right Fig. II. Line which passes through C, the middle of AB, and being produced Bisefts the part within the Curve of all Parallels to the Tangent at the Vertex of the Curve, and the Lines so Bisefted are called Ordinates to that Diameter. Thus, FY is a Diameter, rb , bz , are Ordinates being Parallel to the Tangent FT, which touches the Curve in F, the Vertex of the Diameter.
3. The point of Concourse of all the Diameters (as C) is called the Center.
4. That produced Diameter to which the Ordinates stand at right Angles (as AB) is called the Axe.
5. The common Interfection of the Diameter produc'd and the Ordinate (as G, or b ,) is called the point of Application.
6. That part of the Diameter produced, which is Intercepted between the Vertex and point of Application, is called the Abscissa, as BG or Fb.
7. If, on (B) the Vertex of the Axe, a Per- Fig. I. pendicular to the Axe be drawn and continued both ways, and then if, from the Center C, with the Radius CK, you Intersect that Perpendicular

pendicular in the points D and E, right-Lines passing through the points CE, CD, are called Asymptotes; and the Perpendicular intercepted between them (as ED,) is call'd the Conjugate Axe.

P R O P. I.

AS the Square of the Transverse Axe, is to the Square of the Conjugate Axe; so is the Rectangle of the Transverse added to the Abscissa, into the Abscissa, to the Square of the Ordinate applied to that Abscissa; that is, ABq. DEq. BG \times AG. GFq.

Fig. III.

D E M O N S T.

Put $AC = \frac{1}{2}t$, $AE = \frac{1}{2}c$, $CG = x$, $CK = CH = b$, $GF = y$; then $GK = x \text{ or } b$, and $KH = 2b$, and let $FK = z$; then (by the Genesis) $FH = t + z$, and $AEq + ACq = (CEq =) CKq$; that is, $\frac{1}{4}c^2 + \frac{1}{4}t^2 = b^2$ by 47. E. 1; and (by 12. and 13. E. 2.) $HFq = KHq + FKq \pm 2KH \times GK$; that is, $t^2 + 2tz + z^2 =$

$$z^2 + 4b^2 + 4bx - 4b^2 \therefore z = \frac{4bx - t^2}{2t}; \text{ and}$$

by Squaring both sides, $\frac{16b^2x^2 + t^4 - 8bt^2x}{4t^2}$

$= (z^2 =) y^2 + x^2 - 2bx + b^2$; which being clear'd of Fractions and contradictory Terms, will become $16b^2x^2 + t^4 = 4t^2y^2 + 4t^2x^2 + 4t^2b^2$; and if, for $16b^2$ and $4b^2$ in this Equation, we substitute their respective values in the first, and throw away contradictory Terms, and divide by 4, we shall have $t^2y^2 = c^2x^2 - \frac{1}{4}t^2c^2$, which

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which reduced to an Analogy, gives $t^2. c^2 :: x + \frac{1}{2}t \times x - \frac{1}{2}t. y^2$. or, $ABq.DEq :: BG \times AG$.
FGq. Q. E. D.

COROL.

Let the Transverse and Conjugate Axes be represented by t , and c , any Abscissa and it's Ordinate by x , and y , then by this Theorem $t^2. c^2 :: t + x \times x. y^2$ $\therefore t^2 y^2 = c^2 t x + c^2 x^2$, which is the Equation of the Curve.

Definition. A third Proportional to the Transverse and Conjugate Axes; is called the Parameter of the Axis, that is if p , be put for the Parameter, $t. c :: c. p \therefore tp = c^2$.

PROP. II.

AS the Transverse Axis, is to the Parameter of the Axis, so is the Transverse added to any Abscissa, into that Abscissa, to the Square of the Ordinate apply'd to that Abscissa. that is, $t. p :: t + x \times x. y^2$.

DEMONST.

By the Construction of the Parameter, $tp = c^2$; and if, in the Equation of the Curve, we substitute tp , for cc , we shall have $ty^2 = tp x + px^2$ (which is the Equation of the Curve in the Terms of the Parameter): which being reduced to an Analogy gives, $t. p :: t + x \times x. y^2$.
Q. E. D.

COR.

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COROL.

As the Rectangle of the Transverse added to any Abscissa into that Abscissa, is to the Square of the Ordinate applied to that Abscissa; so is the Rectangle of the Transverse added to any other Abscissa into that Abscissa, to the Square of the Ordinate apply'd to that Abscissa; For (by this Prop.) $t + x \times x.y^2 :: (t.p ::) t + X \times X.Y^2$.

PROP. III.

A Shalf the Transverse Axe, is to the Sum of the Transverse and Focal distance, so is the Focal distance, to half the Parameter of the Axe. that is, (by putting q , for the Focal distance) $\frac{1}{2}t. \frac{1}{2}t + q :: q. \frac{1}{2}p$.

DEMONST.

Fig. III. $CK(=CE) - AC = AK$, that is, $\sqrt{\frac{1}{4}t^2 + \frac{1}{4}c^2} - \frac{1}{2}t = q$; but (by the 2d.) $\frac{1}{4}c^2 = \frac{1}{4}pt$, $\therefore \sqrt{\frac{1}{4}t^2 + \frac{1}{4}tp} - \frac{1}{2}t = q$; and $\frac{1}{4}tp = tq + q^2$. *i. e.* $\frac{1}{2}t. t + q :: q. \frac{1}{2}p$. Q. E. D.

PROP. IV.

THE Parameter of the Axe is equal to double the Ordinate passing through the Focus; that is, (if y , be put for the Ordinate passing through the Focus) $y = \frac{1}{2}p$, or $p = 2y$.

DEMON.

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DEMONST.

By the 2d. (if you put q , for the Focal distance) $t. p :: t + q \times q. y^2$; and (by the 3d.) $t + q \times q = \frac{1}{4}tp \therefore$ (by Substitution) $t. p :: \frac{1}{4}tp. (\frac{1}{4}p^2 =) y^2$ and $\frac{1}{4}p = y. Q. E. D.$

PROP. V.

AS the Sum of the Transverse Axe and it's Parameter, is to the distance between the Foci, so is the distance between the Foci, to the Transverse Axe.

DEMONST.

Put $KH = b$, then $\frac{1}{2}b = (\frac{1}{2}KH = CK = CE)$ $\sqrt{\frac{1}{4}t^2 + \frac{1}{4}c^2}$; and $\frac{1}{4}b^2 = \frac{1}{4}t^2 + \frac{1}{4}c^2$, or $b^2 = t^2 + c^2$. But (by Prop. 2d.) $tp = c^2 \therefore b^2 = t^2 + tp$. that is, $t + p. b :: b. t. Q. E. D.$

PROP. VI.

A Fourth Proportional to the Conjugate Axe, Transverse Axe, and any Ordinate, is a mean Proportional between the Transverse added to the Abscissa, and the Abscissa of that Ordinate.

DEMONST.

Let the fourth Proportional be b ; then, $c. t :: y. b$, and $\frac{ty}{c} = b$. But (by Prop. I.) $t^2. c^2$

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$\therefore \overline{t+x \times x} . y^2$. or $t . c \therefore \sqrt{t+x \times x} . y$, therefore $\sqrt{t+x \times x} = \left(\frac{ty}{c} =\right) b$. Q. E. D.

PROP. VII.

AS the Square of any Ordinate, is to the Rectangle of the Transverse added to the Abscissa into the Abscissa; so is the Square of the Conjugate Axe, to the Square of the Conjugate Axe subducted from the Square of the distance of the Foci.

DEMONST.

Let the distance between the Foci be b , then $c^2 + t^2 = b^2$, and $t^2 = b^2 - c^2$. But (by the 1.) $y^2 . \overline{t+x \times x} \therefore c^2$. ($t^2 =$) $b^2 - c^2$. Q. E. D.

PROP. VIII.

AS the Square of any Ordinate, is to the Rectangle of the Parameter of the Axe into the Abscissa; so is the Square of the Conjugate Axe added to the same Rectangle, to the Square of the Conjugate Axe. *i. e.* $y^2 . px \therefore c^2 + px . c^2$.

DEMONST.

By the Equation of the Curve $t^2 y^2 = c^2 tx + c^2 x^2$; and (by the 2) $\frac{c^2}{p} = t \therefore$ (by Substitution and Expunging,) $c^2 y^2 = px c^2 + p^2 x^2$. that is, $y^2 . px \therefore c^2 + px . c^2$. Q. E. D.

PROP.

PROP. IX.

AS the distance from the Center to the Ordinate drawn from the point of Contact of any Tangent, is to the Abcissa of that Ordinate, so is the Sum of the Transverse and Abcissa, to the Subtangent. that is, CG. AG :: BG. GT. Fig. IV.

DEMONST.

Suppose Fp , an indefinitely small part of the Curve, and produc'd so as to cut the Axe in T ; draw the Ordinate FG , and Parallel to it, pq ; draw Fr Parallel to the Axe, and put $AT = a$, $Fr = qG = n$, and $rp = m$. then $GT = a + x$, $Bq = t + x + n$, $Aq = x + n$, and $pq = y + m$. But $pr.rF :: FG.GT$; that is, $m.n :: y.x + a$ $\therefore n \times \frac{y}{m} = x + a$, and (by Prop. 2d.)

$t.p :: \overline{t+x+n} \times \overline{x+n} \cdot \overline{y+m} \times \overline{y+m}$; also $t.p :: \overline{t+x} \times x \cdot y^2$. whence (from the 1st. Analogy,) $ptx + ptn + px^2 + 2pxn - 2tym = (ty^2 =$ from the 2d. Analogy) $ptx + px^2 \therefore ptn + 2pxn = 2tym$; and $n = \frac{2tym}{pt + 2px}$. But $a + x = n \times \frac{y}{m}$ therefore $a + x = \left(\frac{2tym}{pt + 2px} \times \frac{y}{m} = \frac{2ty^2}{pt + 2px} = \frac{ty^2}{p} \times \frac{2}{t + 2x} = tx + x^2 \times \frac{2}{t + 2x} = \frac{2tx + 2x^2}{t + 2x} = \right) \frac{tx + x^2}{\frac{1}{2}t + x}$; that is, $\frac{1}{2}t + x. x :: t + x. a + x$. or CG. AG :: BG. GT. Q. E. D.

PROP.

PROP. X.

AS half the Transverse added to the Abscissa of the Ordinate from the point of Contact, is to half the Transverse Axe, so is half the Transverse Axe, to the distance (in the Axe produc'd) from the Center to the Intersection of the Tangent. that is, CG. CA :: CA. CT.

DEMONST.

$$\begin{aligned} \text{CT} &= \text{CG} - \text{GT}. \text{ But } \text{CT} = \frac{1}{2}t - a, \text{ CG} \\ &= \frac{1}{2}t + x; \text{ and (by the last) } \text{GT} = \frac{tx + x^2}{\frac{1}{2}t + x} \therefore \\ \frac{1}{2}t - a &= \left(\frac{1}{2}t + x - \frac{tx + x^2}{\frac{1}{2}t + x} \right) = \frac{\frac{1}{4}t^2}{\frac{1}{2}t + x} \text{ and} \\ \frac{1}{2}t + x. \frac{1}{2}t &:: \frac{1}{2}t. \frac{1}{2}t - a, \text{ or CG. CA} :: \text{CA. CT.} \\ \text{Q. E. D.} \end{aligned}$$

PROP. XI.

AS half the Transverse added to the Abscissa of the Ordinate drawn from the point of Contact, is to half the Transverse Axe; so is the Abscissa, to the distance from the Vertex to the Intersection of the Tangent; that is, CG. AC :: AG. AT.

DEMONST.

$$\begin{aligned} \text{By the last, } \frac{1}{2}t - a &= \frac{\frac{1}{4}t^2}{\frac{1}{2}t + x}, \text{ therefore } a = \\ &= \frac{\frac{1}{2}tx}{\frac{1}{2}t + x}; \text{ and } \frac{1}{2}t + x. \frac{1}{2}t :: x. a; \text{ or, CG. AC} :: \\ \text{AG. AT. Q. E. D.} \end{aligned}$$

PROP.

PROP. XII.

AS half the Transverse added to the Abscissa of the Ordinate from the point of Contact, is to half the Transverse; so is the Transverse added to the Abscissa, to the Transverse less by the External part; that is, $CG. CA :: BG. BT$. Fig. IV.

DEMONST.

By the II. $a = \frac{\frac{1}{2}tx}{\frac{1}{2}t+x}$; $\therefore t - a = (t - \frac{\frac{1}{2}tx}{\frac{1}{2}t+x}) = \frac{\frac{1}{2}t^2 + \frac{1}{2}tx}{\frac{1}{2}t+x}$, and $\frac{1}{2}t+x. \frac{1}{2}t :: t+x. t-a$; or, $CG. CA :: BG. BT$. *Q. E. D.*

PROP. XIII.

AS the Transverse added to the Abscissa of the Ordinate from the point of Contact, is to the Transverse less by the External part; so is the Abscissa, to the External part. that is, $BG. BT :: GA. AT$.

DEMONST.

By the II. $\frac{1}{2}t+x. \frac{1}{2}t :: x. a$. and (by the 12.) $\frac{1}{2}t+x. \frac{1}{2}t :: t+x. t-a$ \therefore (by Equality) $t+x. t-a :: x. a$; or, $BG. BT :: AG. AT$. *Q. E. D.*

PROP. XIV.

AS half the Transverse less by the External part, is to half the Transverse; so is the External part, to the Abscissa of the Ordinate from

from the point of Contact; that is, CT. CA
 \therefore AT. AG.

DEMONST.

By the 11th. $a = \frac{\frac{1}{2}tx}{\frac{1}{2}t + x}$; whence $x = \frac{\frac{1}{2}ta}{\frac{1}{2}t - a}$, and $\frac{1}{2}t - a. \frac{1}{2}t \therefore a. x$; or, CT. CA
 \therefore TA. AG. $\mathcal{Q}. E. D.$

PROP. XV.

AS half the Transverse less by the External part, is to the Transverse less by the External part; so is the External part, to the Abscissa of the Ordinate from the point of Contact added to the External part. that is, CT. BT \therefore AT. GT.

DEMONST.

Fig. IV. By the last, $x = \frac{\frac{1}{2}ta}{\frac{1}{2}t - a} \therefore x + a = \left(a + \frac{\frac{1}{2}ta}{\frac{1}{2}t - a} \right) \frac{ta - a^2}{\frac{1}{2}t - a}$; and $\frac{1}{2}t - a. t - a \therefore a. x + a$; or, CT. BT \therefore AT. GT. $\mathcal{Q}. E. D.$

PROP. XVI.

AS the Transverse Axe added to the Abscissa of the Ordinate from the point of Contact, is to half the Transverse Axe; so is the same Abscissa added to the External part, to the External part; *i. e.* BG. CA \therefore GT. AT.

DEMON.

DEMONST.

By the II. $\frac{1}{2}t + x = \frac{\frac{1}{2}tx}{a}$; $\therefore t + x = (\frac{1}{2}t + \frac{\frac{1}{2}tx}{a}) \frac{\frac{1}{2}ta + \frac{1}{2}tx}{a}$, and $t + x. \frac{1}{2}t :: x + a. a$; or,
BG. CA :: TG. AT. Q. E. D.

PROP. XVII.

AS the Transverse Axe less by the External part, is to half the Transverse Axe; so is the Subtangent, to the Abscissa of the Ordinate drawn from the point of Contact; that is, BT. CA :: TG. AG.

DEMONST.

By the I4. $\frac{1}{2}t - a = \frac{\frac{1}{2}ta}{x}$; $\therefore t - a = (\frac{1}{2}t + \frac{\frac{1}{2}ta}{x}) \frac{\frac{1}{2}tx + \frac{1}{2}ta}{x}$, and, $t - a. \frac{1}{2}t :: x + a. x$;
or, BT. CA :: TG. AG. Q. E. D.

PROP. XVIII.

THE Ratio of the Ordinate drawn from the point of Contact to the Subtangent; is equal to the Ratio compounded of the Ratio's of the distance between the Center and Ordinate, to the Ordinate; and of the Ratio of the Parameter of the Axe, to the Axe. that is,
 $\frac{GF}{GT} = \frac{CG}{FG} \times \frac{p}{t}$.

DEMON.

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DEMONST.

Fig. IV. By the 9. $tx + x^2 = \frac{1}{2}t + x \times x + a$; and, (by the 2d.) $t.p :: (tx + x^2 =) \frac{1}{2}t + x \times x + a.y^2 \therefore ty^2 = p \times \frac{1}{2}t + x \times x + a$; and (if you divide by $x + a$) $\frac{ty^2}{x + a} = p \times \frac{\frac{1}{2}t + x}{x + a}$, and (again if you divide by ty) $\frac{y}{x + a} = \left(\frac{p \times \frac{1}{2}t + x}{ty} = \right) \frac{\frac{1}{2}t + x}{y} \times \frac{p}{t}$; or, $\frac{GF}{GT} = \frac{CG}{FG} \times \frac{p}{t}$. Q. E. D.

PROP. XIX.

IF, from the Vertices of the opposite Sections, and from the Center, Perpendiculars be drawn to the Axe, and cut any Tangent, and also an Ordinate be drawn from the point of Contact, then these four Lines shall be Proportional; that is, $BO.CP :: FG.AQ$.

DEMONST.

By the 15. $TB.TC :: TG.AT \therefore$ (by 4 Eu. 6.) $BO.CP :: FG.AQ$. Q. E. D.

COROL.

Hence, $BO \times AQ = CP \times FG$.

PROP. XX.

IF Perpendiculars to the Axe be drawn from the Vertexes of the opposite Sections, and cut any Tangent, the Rectangle of these Perpendiculars

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pendiculars shall be equal to the Rectangle of the greatest and least distance of either Focus from the Vertex; that is, $BO \times AQ = KA \times KB = AH \times BH$.

DEMONST.

Let $AQ = m$, $BO = n$, and AK or $BH = q$; then (by Similar Δ 's,) $m.y :: (a.x + a ::$ by the 16) $\frac{1}{2}t.t + x$; also $n.y :: (t - a.x + a ::$ by 17) $\frac{1}{2}t.x :: m = \frac{\frac{1}{2}ty}{t+x}$; and $n = \frac{\frac{1}{2}ty}{x}$ and (these being Multiplied,) $mn = \frac{\frac{1}{4}t^2y^2}{tx + x^2} :: mn. \frac{1}{4}t^2 :: (y^2.tx + x^2 ::$ by 2d.) $p.t$ and $mn = (\frac{1}{2}pt =$ by the 3d.) $t+q \times q$. or $BO \times AQ = AK \times KB = AH \times BH$.

LEMMA.

If, on the Extremities of any Chord Line Fig. V.
AB, Perpendiculars as BQ, AD, be drawn, and if any right-Line as DQ pass through the Center, and cut these Perpendiculars, then the External parts OQ, PD, of that Line shall be equal; and the Rectangle of the Perpendiculars shall be equal to the Rectangle of the Secant QP into the External part QO.

DEMONST.

Because the Angle A is right, $\therefore BN$ is a Diameter and passes through the Center C, and ΔCBQ , is Similar to the $\Delta CND \therefore CB.CN :: CQ.CD :: BQ.ND$. But $CB = CN \therefore CQ = CD, BQ = ND$, and $QP = DO$. but (by 36.
K E. 3.

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E. 3.) $DA \times (DN =) BQ = DO \times DP = QP \times QO$. Q. E. D.

PROP. XXI.

Fig. VI. IF, from the Intersections (P, S,) of a Circle drawn on the Transverse, with any Tangent, Perpendiculars (as Ph, Sk) to the Tangent be drawn, I say they will Intersect the Transverse Axe produc'd in the Focal points K and H.

DEMONST.

The Δ 's, TOB, TPb, TAQ, and TS*k* having the Angles at T, common, and each a right Angle, are Similar, $\therefore BO.Pb :: Sk.AQ$ and $BO \times AQ = (Ph \times Sk =$ by the preceding Lemma) $hA \times hB$, or, $kB \times kA$. But (by 20.) $BO \times AQ = HA \times HB$, or, $KA \times KB$, \therefore the points K, *k*; and H, *h* are Coincident. Q. E. D.

COROL.

$KS \times PH = \frac{1}{4}pt.$ because $HA \times HB = \frac{1}{4}pt$ by 3d.

PROP. XXII.

Fig. VII. IF, from any part of the Curve, Lines be drawn to the Foci, and the Angle formed by those Lines be Bisected, then the Bisecting Line will be a Tangent to the Curve in the Angular point.

DEMON.

DEMONST.

Take $FX = FK$, then (because by Hypo-
thesis $\angle HFT = \angle TFK$) if you take any point Fig. VII.
 S , in the Line FT , the Line $KS = SX$, by 4.
E. 1. Draw SH , then $AB (=HX) + (SX$
 $=) SK$ is greater than $SH \therefore$ the point S , is
without the Curve; for, if it were in the Curve,
 $AB + SK$ would be equal to SH , by the Ge-
nesis.

COROL.

Hence Lines drawn from the Foci to the point
of Contact, make with the Tangent equal An-
gles.

PROP. XXIII.

A Right Line Perpendicular to any Tangent
at the point of Contact, Bisechs the Angle
made by Lines drawn through the point of Con-
tact, and from the Focus points. that is, if FY , Fig.
VIII.
be Perpendicular to FT , then the $\angle ZFY =$
 $\angle HFY$.

DEMONST.

The $\angle LFY = \angle TFY$ by Hypothesis, and
the $\angle LFZ = (KFT = \text{by 22.}) TFH$. which
being taken from the former, there remains
 $\angle ZFY = \angle HFY$. Q. E. D.

PROP.

P R O P. XXIV.

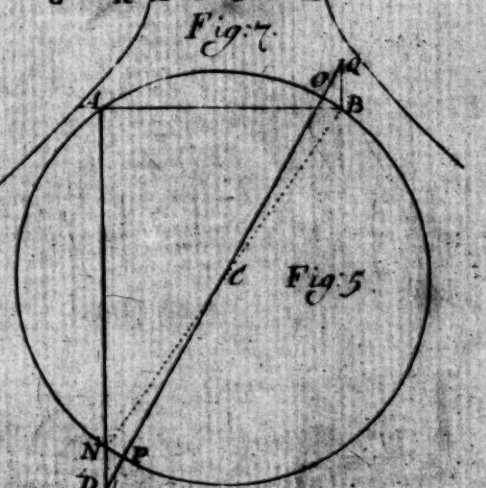
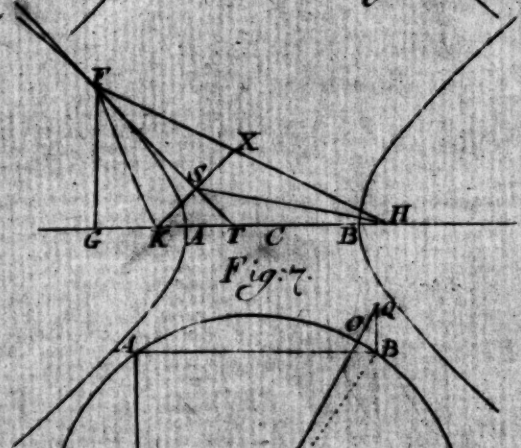
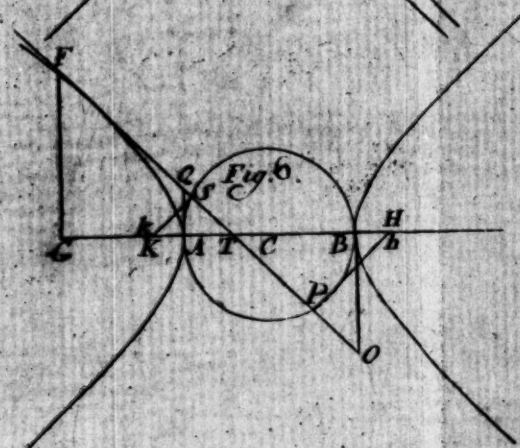
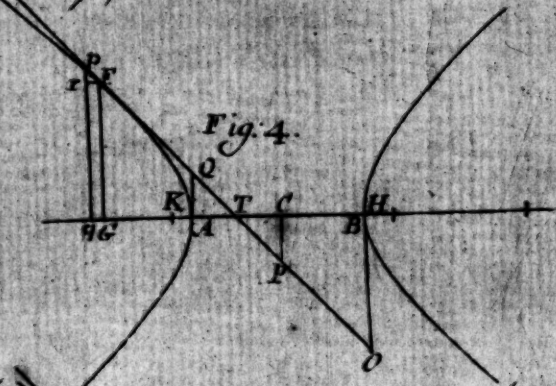
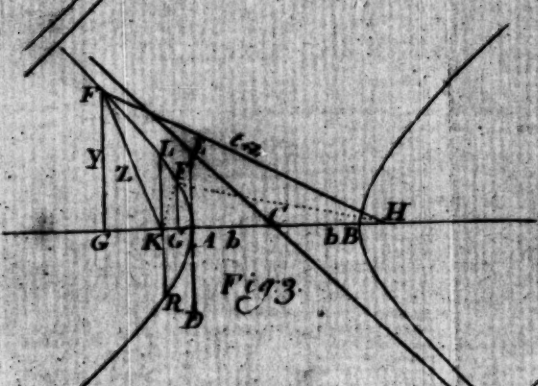
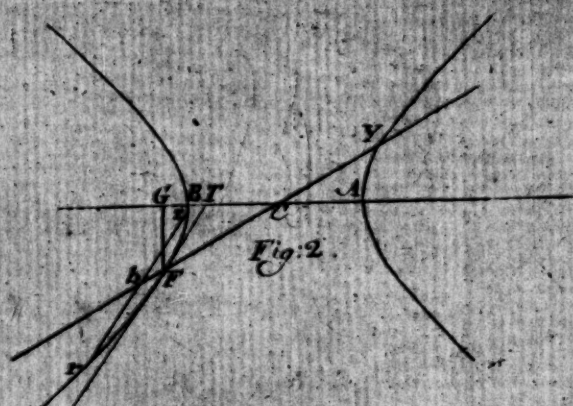
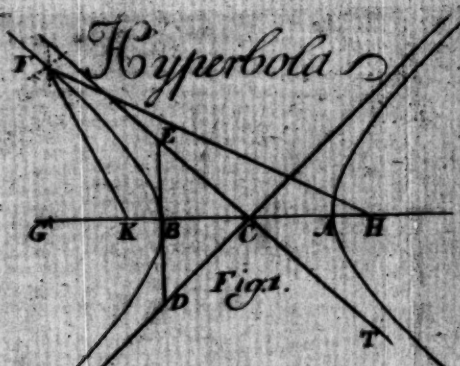
IF, on the Tangent at the point of Contact, a Perpendicular be drawn, and, from the point where that Perpendicular cuts the Axe, two Lines be drawn Perpendicular to the Lines which Connect the Foci to the point of Contact, then the distance on these Lines between the point of Contact and the Perpendiculars, will be equal to half the Parameter of the Axe, *i. e.* $Fq = Fr = \frac{1}{2}p$.

D E M O N S T.

Fig. IX. From the points S, P, where a Circle on the Transverse cuts the Tangent draw Lines to the Foci H and K. which (by the 21) will be Perpendicular to the Tangent \therefore PK, HX, and YF are Parallel; continue HS to X, then (by the 22 d.) $HS = SX$, and $HF = FX$ \therefore $KX = AB = t$, also $\triangle KFY$, is Similar to $\triangle KXH$; and because the $\angle FPK = \angle YqF$, and the Angles PKF, and qFY are the Complements of the $\angle qFL$ \therefore \triangle 's PKF and YFq , are Similar, and $KX : XH :: (KF : FY ::) KP : Fq$, and $KX \times Fq = XH \times KP$ or $\frac{1}{2}KX \times Fq = (\frac{1}{2}XH =) SH \times KP$; that is, $\frac{1}{2}t \times Fq = (SH \times KP = \text{by 21.}) \frac{1}{2}pt$, or $Fq = \frac{1}{2}p$. but (by 26. E. 1.) $Fr = Fq$ \therefore $Fq = Fr = \frac{1}{2}p$. Q. E. D.

P R O P. XXV.

IF Perpendiculars from the Vertices cut any Tangent, the part of the Tangent intercepted between the Intersections shall be the Diameter



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meter of a Circle whose Periphery shall pass through the Foci,

DEMONST.

By the 20. $BO \times AQ = HA \times HB \therefore BO \cdot BH :: AH \cdot AQ$. But $\angle QAH = \angle OBH \therefore$ (by 6. E. 6.) $\triangle AQH$ is Similar to $\triangle OBH$, and $\angle BOH = \angle AHQ$. Also $\angle AQH = \angle BHO$, but $\angle AQH + \angle AHQ =$ a right Angle; $\therefore \angle QHO = (\angle AHQ + \angle BHO =)$ a right Angle, and (by 31. E. 3.) OQ is a Diameter. In like manner QKO may be proved a right Angle. *Q. E. D.*

COROL.

If OQ be Bifected in N , then $NO = HN = NQ$

PROP. XXVI.

IF, from the Remoter Focus, a right Line be drawn to the point of Contact, and in that Line $HX = AB$, and from the other Focus, KX be drawn and cut the Tangent in S , then a right Line drawn from the Center to that Intersection will be equal to half the Transverse Axe; that is, $CS = (\frac{1}{2}AB =) CA$.

DEMONST.

In the \triangle 's KCS, KHX , the $\angle K$ is common; Fig. XI. $KC = CH$, and (by 22) $KS = SX \therefore$ (by 6. E. 6.) the \triangle 's are Similar and CS is \parallel to HX ; also $CS = (\frac{1}{2}HX = \frac{1}{2}AB) = CB = CA$. *Q. E. D.*

PROP.

PROP. XXVII.

IF, from the Remoter Focus a Line be drawn to the point of Contact, and another from the Center \parallel to the Tangent; the distance between the point of Contact, and Intersections of these two Lines is equal to half the Transverse Axe. that is, $FZ = \frac{1}{2}AB$.

DEMONST.

Draw $CS \parallel HF$; then is the Figure $FZCS$
 $a \square \therefore ZF = (CS = \text{by 26.}) AC = \frac{1}{2}AB$,
 $Q. E. D.$

PROP. XXVIII.

IF, within the Curve, Lines be drawn Parallel to any Tangent, they will be Bisected by a Diameter produc'd through the point of Contact. Also $\triangle BCS = \triangle CDr - \triangle pdz = \triangle CVo - \triangle Vpx$.

Fig. XII.

DEMONST.

Fig. XII. Put $dz = y$, $dp = c$, $Cd = n$, $BS = r$, $dr = p$, $Vx = Y$, $CV = g$, $Vo = q$, $Vp = \phi$, and the Abscissas Bd , $BV = x$, X. then,

$$\begin{aligned} 1. \text{ By Similar } \triangle's \frac{y}{c} &= \left(\frac{FG}{GT} = \text{by 18.} \right) \frac{CG}{FG} \\ &\times \frac{p}{t}. \text{ But } \frac{CG}{GF} = \frac{\frac{1}{2}t}{r} \text{ by Similar } \triangle's \therefore \frac{y}{c} = \\ &\frac{\frac{1}{2}t}{r} \times \frac{p}{t}, \text{ Divide by } \frac{p}{t}, \text{ and } \frac{\frac{1}{2}t}{r} = \frac{y}{c} \times \frac{t}{p}; \\ &\text{Multiply} \end{aligned}$$

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Multiply by cry ; and $\frac{1}{2}tcy = ry^2 \times \frac{t}{p}$. But

(by the 2d.) $y^2 \times \frac{t}{p} = tx + x^2 \therefore \frac{1}{2}tcy = r \times \overline{tx + x^2}$, and $rn^2 - \frac{1}{2}tcy = (rn^2 - r \times \overline{tx + x^2}) = r \times \overline{n^2 - tx + x^2}$; and by (6. Eu. 2.) $n^2 - tx + x^2 = \frac{1}{4}t^2 \therefore rn^2 - \frac{1}{2}tcy = r \times \frac{1}{4}t^2$, Divide by $\frac{1}{2}t$; and $\frac{rn^2}{\frac{1}{2}t} - cy = r \times \frac{1}{2}t$. By Similar Δ 's

$\frac{1}{2}t. r :: n. p = \frac{rn}{\frac{1}{2}t} \therefore$ (by Substitution,) $np - cy = r \times \frac{1}{2}t$; or $Cd \times dr - dp \times dz = BS \times BC$. that is, $\Delta Cdr = \Delta pdz = \Delta BCS$.

2. By Similar Δ 's $\frac{Y}{b} = (\frac{FG}{GT} = \text{by 18.}) \frac{CG}{FG} \times \frac{p}{t}$. By Similar Δ 's $\frac{CG}{GF} = \frac{g}{q} \therefore \frac{Y}{b} = \frac{g}{q} \times \frac{p}{t}$. Divide by $\frac{p}{t}$, and $\frac{g}{q} = \frac{Y}{b} \times \frac{t}{p}$; Multiply by bqY , and $gbY = qY^2 \times \frac{t}{p}$; but (by

Prop. 2.) $Y^2 \times \frac{t}{p} = tX + X^2$ therefore $gbY = q \times tX + X^2$ and $qg^2 - gbY = (qg^2 - q \times tX + X^2) = q \times \overline{g^2 - tX + X^2}$. but (by 6. Eu. 2d.) $g^2 - tX + X^2 = \frac{1}{4}t^2$; $\therefore qg^2 - gbY = q \times \frac{1}{4}t^2$ Divide by g , and $qg - bY = \frac{q \times \frac{1}{4}t^2}{g}$

but (by Similar Δ 's) $g. q :: \frac{1}{2}t. r = \frac{q \times \frac{1}{2}t}{g} \therefore$ (by Substitution) $qg - bY = r \times \frac{1}{2}t$, or $Vo \times CV - Vp \times Vx = BC \times BS$; that is, $\Delta CVo = \Delta Vpx = \Delta BCS =$ (by the former part) ΔCdr

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$-\Delta pdz$ and (by Transposition) $\Delta CVo - \Delta Cdr = \Delta Vpx - \Delta pdz$.

3. From both sides of the last Equation take the Figure $dzboV$, and there remains the $\Delta obx =$ and Similar $\Delta b zr \therefore xb = bz$. Q. E. D.

PROP. XXIX.

THE $\Delta BSC = \Delta CFT$; also the Trapezium $dBSr = \Delta pdz$ and $\Delta b zr =$ Trapezium $bFTp$ and $FT + bp \times bF = zb \times rb$.

DEMONST.

From Similar Δ 's $BS.FG :: (BC.GC ::$ by 10.) $CT.BC \therefore BS \times BC = FG \times CT$; or $\Delta BSC = \Delta CFT =$ (by 28.) $\Delta Cdr - \Delta pdz = \Delta CVo - \Delta Vpx \therefore$ (by Transposition) $\Delta BCS + \Delta pdz = \Delta Cdr$; from each side take ΔBCS ; then there remains $\Delta pdz =$ Trapezium $dBSr$; and (by Transposing the 1st. Equations) $\Delta CFT + \Delta Vpx = \Delta CVo$, from each side take $\Delta CFT +$ Trapezium $pboV$, and there remains $(\Delta obx =) \Delta b zr =$ Trapezium $bFTp$, and (by Lemma to Prop. 11. of the Parabola) $FT + bp \times Fb = zb \times br$. Q. E. D.

Definition. Let $FS.FQ :: br.bz :: 2FT$. P , the Parameter of the Diameter FY . then, $P = \frac{bz \times 2FT}{br}$; and,

PROP. XXX.

Fig. XII. **A** Sany Diameter is to it's Parameter (so obtained) so is the Rectangle of the Diameter added to the Abscissa into the Abscissa, to the

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the Square of the Ordinate of that Abscissa. that is, if you put $D = FY$, $x = Fb$, and $y = bz$, or bx , it will be $D. P :: \overline{D+x} \times x. y^2$.

DEMONST.

By the *Definition* $P = \frac{y \cdot 2FT}{br}$ therefore $P \times \frac{\overline{D+x} \times x}{D} = \frac{y \times 2FT}{br} \times \frac{\overline{D+x} \times x}{D} = \frac{y}{br} \times \overline{D+x} \times x \times \frac{2FT}{D}$. But $\frac{2FT}{D} = \left(\frac{FT}{\frac{1}{2}D} = \right)$ by Similar Δ 's $\frac{Tn}{(np \Rightarrow) x}$ therefore (by Substitution) $\frac{P \times \overline{D+x} \times x}{D} = \left(\frac{y}{br} \times \overline{D+x} \times Tn \Rightarrow \right)$ (by Lemma to Prop. 36. of the Ellipse) $\frac{y}{br} \times y \times br = y^2 \therefore D. P :: \overline{D+x} \times x. y^2$. Q. E. D.

PROP. XXXI.

IF a Tangent cut any Diameter, and, if, from the point of Contact, an Ordinate be drawn to that Diameter; then, as the Semi-Diameter added, to the Abscissa is to the Abscissa, so is the Diameter added to the Abscissa, to the Subtangent on that Diameter; that is, $CP. FP :: YP. PT$.

DEMONST.

Let QG be an indefinitely small part of the Curve, and produced to cut the Diameter in T ; Fig. XIII.
L draw

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draw the Ordinate GP, and, Parallel to it, QO; draw Gr Parallel to (YO) the Diameter continued; and put $Gr = n$, $Qr = m$, and $FT = a$. then $YP = D + x$, $YO = D + x + n$, $OF = x + n$, $QO = y + m$, and $PT = x + a$; and (by Similar Δ 's) $m : n :: y : x + a \therefore x + a = n \times \frac{y}{m}$. But (by Prop. 30.) $D.P :: \overline{D + x + n} \times \overline{x + n} : \overline{y + m} \times \overline{y + m}$; and $D.P :: \overline{D + x} \times \overline{x} : \overline{y^2}$. and reducing the first Analogy into an Equation, we shall have $PDx + PDn + Px^2 + 2Pxn - 2Dym = (Dy^2 = \text{in the 2d. Analogy}) PDx + Px^2 \therefore PDn + 2Pxn = 2Dym$, and $n = \frac{2Dym}{PD + 2Px}$. But $x + a = (n \times \frac{y}{m} \therefore x + a = (\frac{2Dym}{PD + 2Px} \times \frac{y}{m} = \frac{2Dy^2}{PD + 2Px} = \frac{Dy^2}{P} \times \frac{2}{D + 2x} = Dx + x^2 \times \frac{2}{D + 2x} = \frac{2Dx + 2x^2}{D + 2x} =) \frac{Dx + x^2}{\frac{1}{2}D + x} \therefore \frac{1}{2}D + x : x :: D + x : x + a$; or, CP.FP :: YP.PT. Q. E. D.

PROP. XXXII.

THE same things being supposed as before, as the Semi-Diameter added to the Abscissa, is to the Semi-Diameter, so is the Semi-Diameter, to the Semi-Diameter less by the External part. *i. e.* CP. CF :: CF. CT.

DEMON.

DEMONST.

$CP - PT = CT$. But $CP = \frac{1}{2}D + x$, $CT = \frac{1}{2}D - a$; and (by Prop. 31.) $PT = \frac{Dx + x^2}{\frac{1}{2}D + x}$
 $\therefore \frac{1}{2}D + x - \frac{Dx + x^2}{\frac{1}{2}D + x}$, or $\frac{\frac{1}{4}D^2}{\frac{1}{2}D + x} = \frac{1}{2}D - a$,
 that is, $\frac{1}{2}D + x : \frac{1}{2}D :: \frac{1}{2}D : \frac{1}{2}D - a$; or $CP : CF :: CF : CT$. Q. E. D.

PROP. XXXIII.

AS the Semi-Diameter added to the Abscissa, is to the Semi-Diameter; so is the Abscissa, to the External part; that is, $CP : CF :: PF : FT$.

DEMONST.

By Prop. 32. $\frac{\frac{1}{4}D^2}{\frac{1}{2}D + x} = \frac{1}{2}D - a \therefore \frac{1}{4}D^2 = \frac{\text{Fig. XIII.}}{\frac{1}{2}D + x} \cdot (\frac{1}{2}D - a)$
 $\frac{1}{4}D^2 + \frac{1}{2}Dx - \frac{1}{2}Da - xa$, and $\frac{1}{2}Da + xa = \frac{1}{2}Dx$
 $\therefore \frac{1}{2}D + x : \frac{1}{2}D :: x : a$; or, $CP : CF :: PF : FT$. Q. E. D.

PROP. XXXIV.

AS the Semi-Diameter added to the Abscissa, is to the Semi-Diameter; so is the Diameter added to the Abscissa, to the Diameter less by the External part; that is, $CP : CF :: YP : YT$.

PROP.

DEMONST.

By Prop. 33. $a = \frac{\frac{1}{2}Dx}{\frac{1}{2}D+x} \therefore D-a = (D - \frac{\frac{1}{2}Dx}{\frac{1}{2}D+x} =) \frac{\frac{1}{2}D^2 + \frac{1}{2}Dx}{\frac{1}{2}D+x}$ and $\frac{1}{2}D+x : \frac{1}{2}D :: D+x : D-a$; or, CP. CF :: YP. YT. Q. E. D.

PROP. XXXV.

AS the Diameter added to the Abscissa, is to the Diameter less by the External part, so is the Abscissa, to the External part; *i. e.* YP. YT :: PF. FT.

DEMONST.

By Prop. 33. $\frac{1}{2}D+x : \frac{1}{2}D :: x : a$. and (by the 34.) $\frac{1}{2}D+x : \frac{1}{2}D :: D+x : D-a$ \therefore (by Equality) $D+x : D-a :: x : a$; or, YP. YT :: PF. FT. Q. E. D.

PROP. XXXVI.

AS the Semi-Diameter less by the External part, is to the Semi-Diameter; so is the External part, to the Abscissa. *i. e.* CT. CF :: FT. FP.

DEMONST.

By Prop. 32. $\frac{\frac{1}{2}D^2}{\frac{1}{2}D+x} = \frac{1}{2}D-a \therefore \frac{1}{2}D^2 = \frac{1}{2}D^2 - \frac{1}{2}Da + \frac{1}{2}Dx - xa$; and $\frac{1}{2}Dx - xa = \frac{1}{2}D$

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$\frac{1}{2}D a$. *i. e.* $\frac{1}{2}D - a$. $\frac{1}{2}D :: a. x$; or, CT. CF :: FT. FP; \mathcal{Q} . E. \mathcal{D} .

PROP. XXXVII.

AS the Semi-Diameter less by the External part, is to the Diameter less by the External part; so is the External part, to the Subtangent. *i. e.* CT. YT :: FT. PT.

DEMONST.

By Prop. 36. $\frac{1}{2}D x - xa = \frac{1}{2}D a \therefore x = \frac{\frac{1}{2}D a}{\frac{1}{2}D - a}$; and $a + x = \left(a + \frac{\frac{1}{2}D a}{\frac{1}{2}D - a} = \right)$ Fig. XIII.
 $\frac{D a - a a}{\frac{1}{2}D - a}$ and $\frac{1}{2}D - a. D - a :: a. x + a$; or CT. YT :: FT. PT. \mathcal{Q} . E. \mathcal{D} .

PROP. XXXVIII.

AS the Semi-Diameter less by the External part, is to the Semi-Diameter; so is the Diameter less by the External part, to the Diameter added to the Abcissa. *i. e.* CT. CF :: YT. YP.

DEMONST.

By Prop. 37. $x = \frac{\frac{1}{2}D a}{\frac{1}{2}D - a} \therefore D + x = \left(D + \frac{\frac{1}{2}D a}{\frac{1}{2}D - a} = \right) \frac{\frac{1}{2}D^2 - \frac{1}{2}D a}{\frac{1}{2}D - a}$. and $\frac{1}{2}D - a. \frac{1}{2}D :: D - a. D + x$; or, CT. CF :: YT. YP. \mathcal{Q} . E. \mathcal{D} .

PROP.

P R O P. XXXIX.

IF any Ordinate to the Axe (as Vx) be continued to (N, in) the Focal Tangent (TO) then the distance (VN) from the Axe to that point in the Tangent, shall be equal to (Kx) the distance from the Focus to the extremity of that Ordinate.

D E M O N S T.

Fig. XIV. Put $CK = b$, $CB = c$, $CV = d$, then $AK = b + c$, $BK = b - c$, $VK = d \cap b$, $BV = d - c$, and $AV = d + c$, then,

1. The point K being the Focus by Prop. 4. $KL =$ half the Parameter of the Axe, and (by Prop 3.) $CB. AK :: KB. KL$, or, $c. b + c :: b - c. \frac{b^2 - c^2}{c} = (KL =) \frac{1}{2}p$. Also (by Prop.

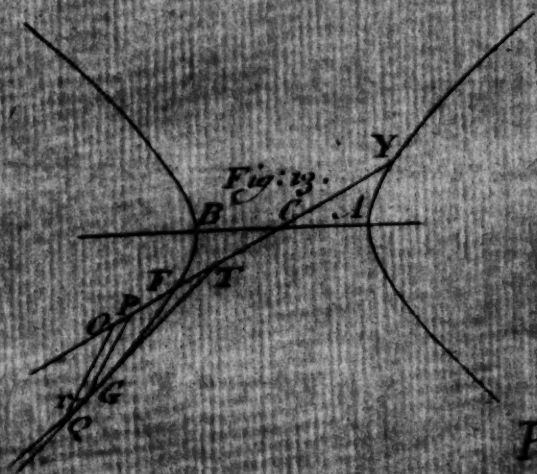
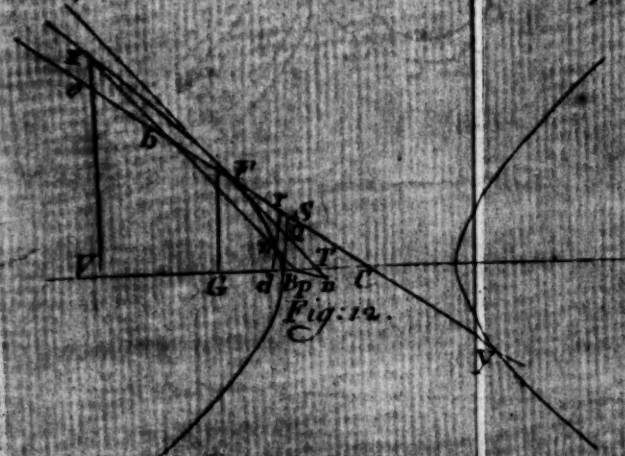
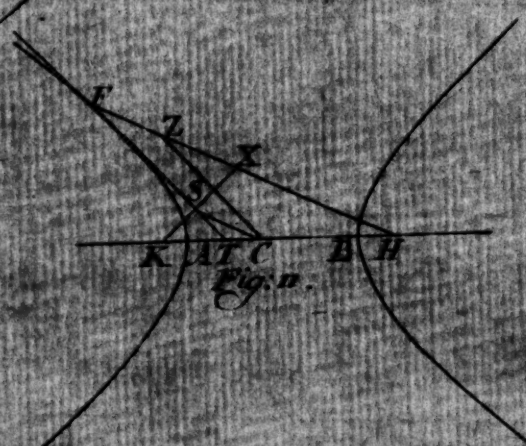
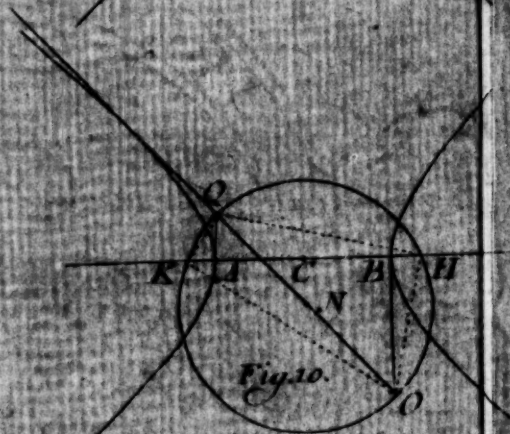
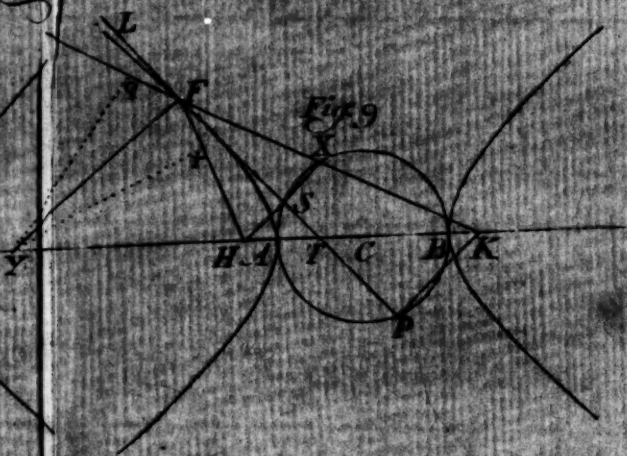
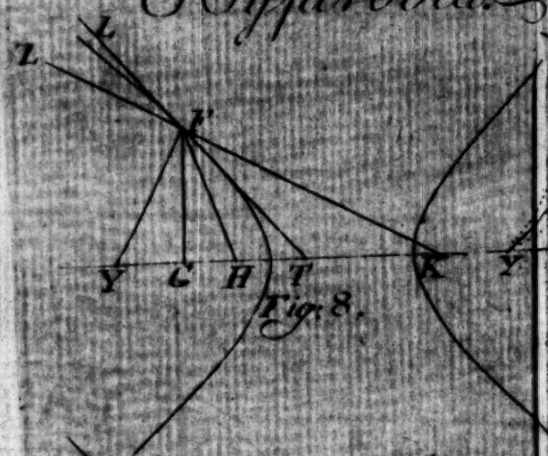
10.) $CK. CB :: CB. CT$; or, $b. c :: c. \frac{c^2}{b} = CT$. But $CK - CT = KT$. i. e. $b - \frac{c^2}{b} = \frac{b^2 - c^2}{b} = KT$, and $CV - CT = VT$, or,

$d - \frac{c^2}{b} = \frac{db - c^2}{b} = VT$; and (by Similar Δ 's) $KT. KL :: VT. VN$, or, $\frac{b^2 - c^2}{b} \cdot \frac{b^2 - c^2}{c} :: \frac{bd - c^2}{b} \cdot \frac{bd - c^2}{c} = VN$.

2. By Prop. 2d. $CB. KL :: AV \times VB. Vxq$. or, $c. \frac{b^2 - c^2}{c} :: d^2 - c^2. \frac{b^2 d^2 - b^2 c^2 - c^2 d^2 + c^4}{c^2}$

=

Hyperbola.



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$= Vxq.$ and $VKq, = d^2 - 2db + b^2$. But
(by 47. E. 1.) $VKq + Vxq = Kxq$, or,
 $\frac{c^2 - 2c^2bd + b^2d^2}{c^2} = Kxq$. and by extracting

the Square Root, $Kx = \left(\frac{bd - c^2}{c} \right) =$ by the
first part) VN . *Q. E. D.*

PROP. XL.

IF Perpendiculars be drawn from the Vertices
to the Focal Tangent, then these Perpendi- Fig.
XIV.
culars shall be equal to the distance (in the Axe)
from each Vertex to it's adjacent Focus respec-
tively; that is, $AO = AK$, and $BQ = BK$.

DEMONST.

By the 20. $AO \times BQ = AK \times KB \therefore AO.$
 $AK :: KB. BQ$. But (by Prop. 39.) $BQ =$
 $BK \therefore AO = AK$. *Q. E. D.*

PROP. XLI.

IF, through the point of Contact of the Focal
Tangent, a right Line be drawn to the Ver- Fig.
XIV.
tex, and any Ordinate be produced to the Tan-
gent, and cut that Line, then the distance be-
tween the Tangent and Intersection of these
Lines is equal to the distance (in the Axe) from
the Focus to the Application of the Ordinate;
that is, $DN = KV$.

DEMON.

DEMONST.

From Similar Δ 's AO. DN :: (LO. LN :: AL. LD ::) AK. KV; But AO = AK by the 40. \therefore DN = KV. Q. E. D.

Of the HYPERBOLIC ASYMPTOTES.

PROP. XLII.

Fig. XV. **I**F any Ordinate to the Axe be continued both ways to the Asymptotes, (as NGP) then, the Square of the Semi-Conjugate Axe, (BE) will be equal to the Rectangle of the greatest and least distance of either extremity of that Line from the Curve; that is, $BEq = NS \times SP = Pr \times rN$.

DEMONST.

Let $NG = PG = b$, and the other Symbols as usual; then $CG = \frac{1}{2}t + x$, and (by Similar Δ 's) $CBq. BEq :: CGq. GNq$. i. e. $\frac{1}{4}t^2. (\frac{1}{4}c^2 =) \frac{1}{4}tp :: \frac{1}{4}t^2 + tx + x^2. b^2 \therefore b^2 = \frac{p}{t} \times \frac{1}{4}t^2 + tx + x^2$. But (by Prop. 2d.) $t. p :: tx + x^2. y^2 \therefore y^2 = \frac{p}{t} \times tx + x^2$, and $b^2 - y^2 = (\frac{p}{t} \times \frac{1}{4}t^2 = \frac{1}{2}pt =) \frac{1}{4}c^2$. Also $b + y. \frac{1}{2}c :: \frac{1}{2}c. b - y$, or NS. EB :: EB. SP. Q. E. D.

PROP.

PROP. XLIII.

THE Asymptotes continually approach to the Curve.

DEMONST.

By the 42d. $EB^2 = NS \times SP = Oa \times aY$
 $\therefore NS. Oa :: aY. SP$. But NS is less than Oa ,
 therefore aY , is less than SP , and consequent-
 ly the point Y is nearer to the Curve, than the
 point P . *Q. E. D.*

PROP. XLIV.

IF the Asymptotes and Curve be infinitely produced, they will never Concur.

DEMONST.

From the two first Analogies of Prop. 42. it follows, that $\frac{1}{2}t + x^2. b^2 :: t + x \times x. y^2$. that is, $CGq. GNq :: AG \times BG. Grq$. But (by 6. Eu. 2d.) CGq is greater than $AG \times BG$ $\therefore GNq$ is greater than Grq , and GN greater than Gr , and consequently wherever the point N is taken, it will never touch the Curve.

PROP. XLV.

IF an Ordinate to the Axe be produced both ways to the Asymptotes, then the parts intercepted on each side between the Curve and Asymptotes are equal. *i. e.* $SP = rN$.

M

DEMON.

DEMONST.

From Similar Δ 's BD. BE. $\therefore GP.GN.$ but $BD = BE \therefore GP = GN$, and the Ordinates GS, Gr, being equal, rN will be $= SP$.

Definition. If the Tangent to the Vertex of any Diameter be continued both ways from the point of Contact, with this Condition, that as the Diameter passing through the point of Contact, is to it's Parameter, so is the Square of the Semi-Diameter, to a fourth Proportional, then if the Square Root of that fourth Proportional be set both ways from the Vertex on the Tangent, (as FP, FQ) the extremities will determine the Conjugate Diameter, and if through these extremities, right Lines be drawn from the Center, (as CP, CQ) they shall be Asymptotes.

PROP. XLVI.

Fig. XVI. IF any Ordinate to a Diameter be produced both ways to the Asymptotes, (as mbn) then the Square of the Semi-Conjugate Diameter, will be equal to the Rectangle of the greatest and least distance of either extremity of that Line from the Curve; that is, $FPq = mz \times zn = nr \times rm$.

DEMONST.

Put $bm = r$, $br = y$, $FP = FQ = \frac{1}{2}c$; then (by the *Definition*) $D.P \therefore CFq.FPq$; or $D.P \therefore \frac{1}{4}D^2.\frac{1}{4}c^2 \therefore \frac{1}{4}c^2 = \frac{\frac{1}{4}PD^2}{D} = \frac{1}{4}PD$. But
(from

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(from Similar Δ 's) $CFq. FPq :: Cbq. bmq.$

that is, $\frac{1}{4}D^2. \frac{1}{4}DP :: \frac{1}{2}D + x^2. r^2 \therefore r^2 = \frac{P}{D} \times$

$\frac{1}{4}D^2 + Dx + x^2.$ and (by Prop. 30.) $D. P ::$
 $\overline{D+x} \times x. y^2 \therefore y^2 = \frac{P}{D} \times \overline{D+x} + x^2,$ and $r^2 =$

$y^2 = \left(\frac{P}{D} \times \frac{1}{4}D^2 \right) = \frac{1}{4}c^2.$ Also $r+y. \frac{1}{2}c :: \frac{1}{2}c, / \delta$

$r-y;$ or, $FPq = mz \times zn = rn \times rm,$

PROP. XLVII.

THE Asymptotes drawn through the extremities of any Conjugate Diameter and produced, do continually approach to the Curve.

DEMONST.

By Prop 46. $mz \times mr = (FPq =) wt \times ws \therefore mz. wt. :: ws. mr.$ But $mz,$ is less than $wt \therefore ws,$ is less than $mr,$ and consequently the point $w,$ is nearer the Curve than the point $m.$ Q. E. D.

PROP. XLVIII.

THE Asymptotes produced through the extremities of the Conjugate Diameter will never meet the Curve.

DEMONST.

By the 46. $\frac{1}{4}PD = \frac{1}{4}c^2 = FPq$ and (by Similar Δ 's) $\overline{bw}^2. \overline{Cb}^2 :: (FPq. CFq :: \frac{1}{4}PD. \frac{1}{4}D^2$ Fig. XVI.
 $::) P.D;$ and (by Prop. 30.) $\overline{bs}^2. Yb \times Fb ::$
 $P.D;$

P.D; therefore (by Equality) \overline{bs}^2 . $Yb \times bF :: \overline{bw}^2$. \overline{Cb}^2 . But by (6. Eu. 2d.) \overline{Cb}^2 is greater than $Yb \times bF :: \overline{bw}^2$ is greater than \overline{bs}^2 and bw greater than bs ; consequently wherever the point w , be taken in CP produced, it will be without the Curve. Q. E. D.

PROP. XLIX.

IF Ordinates to any Diameter be produced both ways to the Asymptotes; then the External parts between the Asymptotes and the Curve are equal. *i. e.* $rm = zn$.

DEMONST.

From Similar Δ 's PF.FQ :: bm . bn . But PF = FQ, $\therefore bm = bn$. from which, if you take away the equal Ordinates, there will remain $rm = zn$. Q. E. D.

PROP. L.

Fig.
XVII.

IF the right Line rt , be drawn Parallel to the Diameter FY, then the Square of the Semi-Diameter CF, shall be equal to the Rectangle contained under the greatest and least distance of either extremity of that Line from it's adjacent Asymptote, that is, $CFq = rd \times rp = pt \times td$.

DEMONST.

Put $FQ = PF = c$, $FC = t$, $mr = b$, $rp = d$, $rn = p$, and $rd = q$. then, because Δ
 mrp

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mrp is Similar $\triangle PFC$, and $\triangle nrd$ is Similar $\triangle QFC$, by 4. E. 6. $\frac{b}{d} = \frac{c}{t}$ and $\frac{p}{q} = \frac{c}{t} \therefore$

$\frac{pb}{qd} = \frac{c^2}{t^2}$ or $pb.c^2 :: qd.t^2$. But (by the 46) $bp = c^2 \therefore qd = t^2$, or $CFq = rd \times rp$. Q. E. D.

PROP. LI.

IF a right Line be drawn Parallel to any Diameter and cut the opposite Hyperbolas; then the parts of that Line intercepted between the Curves and Asymptotes are equal, that is, $rp = td$.

DEMONST.

Make the Abscissa, $Yo = Fb$; draw the Ordinate ot , and the Conjugate YR . then, from Similar \triangle 's, $mr.rp :: (PF = FQ.FC :: YR.YC ::) St.td$. But $mr = (zn =) St \therefore rp = td$. Q. E. D.

PROP. LII.

IF, through any two points (L, M) in the Curve, right-Lines (LV, MT) be drawn Parallel to the Asymptotes, then the Rectangles under each of these Lines and the adjacent distance (on the Asymptote) from the Center, shall be equal, that is, $LV \times VC = MT \times TC$. Fig. XVIII.

DEMON.

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DEMONST.

Through the points L, and M, draw the right Line LM, and let $RL = y$, $LV = d$, $RV = p$, $MQ = z$, $QT = x$, $MT = c$, $VC = b$, and $TC = a$. then, because of Parallels, the Δ 's RVL, RCQ, MTQ are Similar. But (by 49.) $y = z$, $\therefore c = p$, and $x = d$, and $c (= p.)$
 $d :: (p + b =) c + b. (a + x =) a + d \therefore ca = db.$ or $LV \times VC = MT \times TC$. Q. E. D.

COROL. I.

Hence if the Lines MT, rs, qt , &c. be drawn Parallel to the Asymptote CR, and the Parallelograms Tm, Sn, to , &c. be inscribed, they will be equal to each other. Because, by the same reasoning, as in this Prop. we may prove each of them equal to the Parallelogram LC.

COROL. II.

Each of the Inscribed Parallelograms Tm, Sn, &c. is equal to the Square of a Right Line (as BS) drawn from the Vertex B, Parallel to the Asymptote CR.

For, (by this Prop.) each of them is equal to $BS (= GC) \times SC$; but (by the Genesis) the $\angle BCG = \angle BCS$, and (from Parallels) $\angle BCG = \angle SBC$, $\therefore \angle BCS = \angle SBC$, and (by the 6. E. I.) $BS = SC$; and consequently each of the Parallelograms Tm, sn, &c. is equal to $BS \times BS$, or BS^2 .

Scholium

PART. III. Of the HYPERBOLA. 95

Scholium. Right Lines drawn from one Asymptote, and Parallel to the other, and terminated by the Curve, (as tq , sr , &c.) are called Ordinates; and the distance of those Lines from the Center (as tC , sC) Abscissas, and a right Line drawn from the Vertex, Parallel to the Asymptote, (as BS) the Parameter of the exterior Hyperbola; and if p , be put for such Parameter, x for the Abscissa, and y for the Ordinate; then (by the last Coroll.) $pp = yx$.

PROP. LIII.

IF, on either of the Asymptotes (as CF) from the Center right-Lines be set off in Continual Proportion (as CD , CE , CF) and if, from the Extremities of these Lines, there be drawn Lines Parallel to the other Asymptote and continued to the Curve, (as DG , EH , FI) they shall likewise be in Continual Proportion; that is, if CD , CE , CF be \div then DG , EH , FI will be \div

Fig.
XIX.

DEMONST.

By the 52. $GD \times DC = HE \times CE$, and $CF \times FI = CE \times EH$. and (by Supposition) $CD \times CF = CE \times CE$. $\therefore DG \cdot EH :: (CE \cdot DC :: CF \cdot CE ::) EH \cdot FI$. Q. E. D.

PROP. LIV.

IF, on either Asymptote there be set off equal parts from the Center, that is, if, right Lines be set off from the Center in continual Arithmetical

metrical Proportion, (as CM, CN, CO, CP, &c.) and from the extremities of these, there be drawn right Lines Parallel to the other Asymptote, and continued to the Curve, (as MR, NS, OT, PV) these shall be in continued Harmonic Proportion.

DEMONST.

$$MR (=CX) \times CM = NS \times CN = OT \times CO = VP \times CP \text{ therefore,}$$

Fig.
XIX.

$$\left. \begin{array}{l} CM.CN :: NS.MR \\ CM.CO :: TO.MR \\ CM.CP :: VP.MR \end{array} \right\} \text{ but } CM = \left\{ \begin{array}{l} \frac{1}{3}CN \\ \frac{1}{3}CO \\ \frac{1}{4}CP \end{array} \right\} \therefore$$

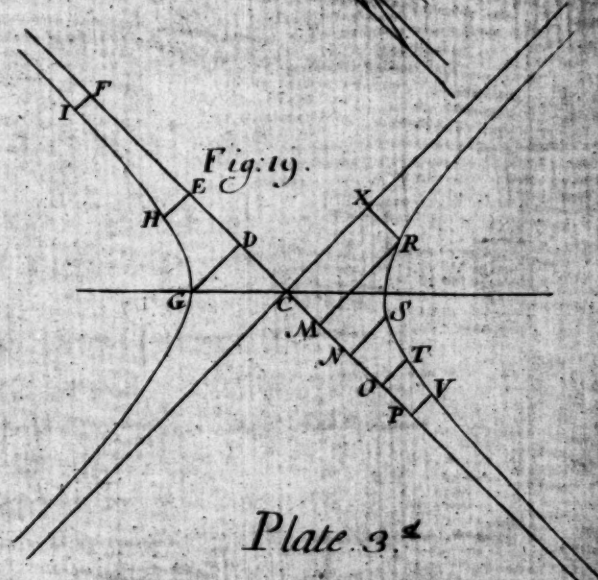
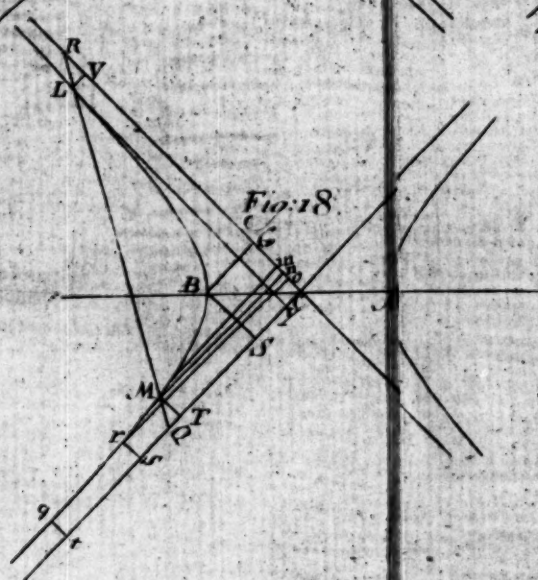
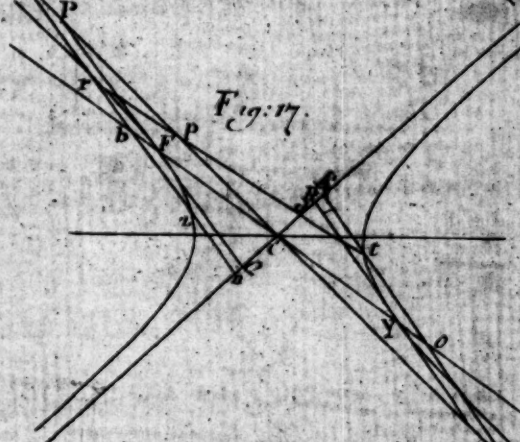
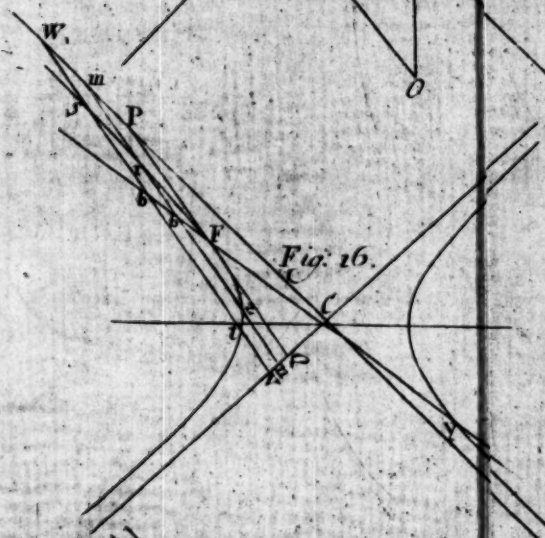
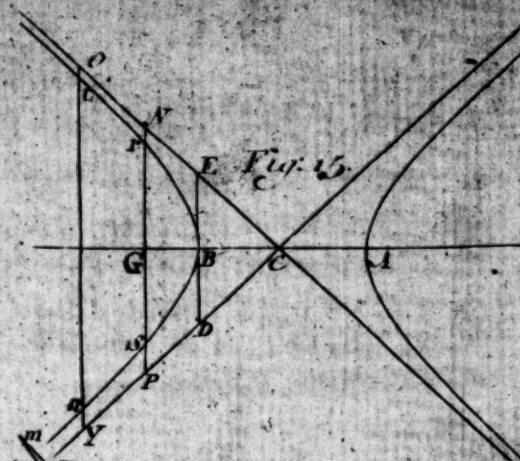
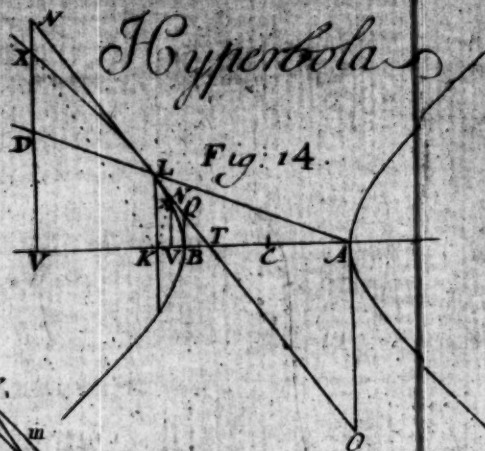
$\left\{ \begin{array}{l} NS \\ TO \\ VP \end{array} \right\} \frac{1}{3} = MR$ and if MR be equal to 1, then $NS = \frac{1}{3}$, $OT = \frac{1}{3}$, $VP = \frac{1}{4}$. which (being the Reciprocals of continued Arithmetical Proportion) are in continued Harmonic Proportion. Q. E. D.

PROP LV.

Fig. XX. IF from the Center on either Asymptote, there be set three Continual Proportionals (as CD, CE, CF) and from their Extremities right Lines be drawn Parallel to the other Asymptote, and continued to the Curve, (as DG, EH, FI) and if on the Curve through the ends of the Extrems (as I, G,) a right Line be drawn as LM, then I say a right Line drawn from the Center through (H) the end of the mean, shall Bisect that Line. that is, CO, Bisects IG in O.

DEMONST.

Hyperbola



DEMONST.

Draw HK Parallel to LM; then (from Similar Δ 's) $EH.KH :: FI.LI$; and $EH.KH :: DG.GL :: EHq.HKq :: FI \times DG.LI \times GL$. But (by the 53.) $EHq = FI \times DG :: HKq = LI \times GL$; and (by the 46.) KH is a Tangent to the point H; and consequently IO = OG, being Parallel to it; is an Ordinate to the Diameter CO. Q. E. D.

PROP. LVI.

IF CD, CE, CF on the Asymptote (and consi- Fig XX.
sequently by the 53 DG, EH, FI) be in continual Proportion; then the Spaces (HEDG, EHFI) between the Curve and Asymptotes on each side of the mean, (EH) to the Extrems (FI and DG) shall be equal.

DEMONST.

1. Through the points I and G, draw the right Line LM; and, through the Center and H, draw the right Line CO; then (by Props. 55 and 49.) $LO = OM ::$ (by 1. Eu. 6.) $\Delta MOC = \Delta OCL$. But the Space OGH = the Space OHI, because each is compos'd of an indefinite Number of equal Ordinates, consequently the Space CHGM = Space CHIL; and from each take away the Δ 's MGP + NHC = Δ 's FLI + HCE, then there remains the Space NHGP = Space EHFI.

2. But $\square CG = \square EN$ by 52. $\therefore \square NG = \square RE$
and consequently the space HEDG = (Space
N HRG

HRG + \square RE = Space HRG + \square NG =
 Space NHGP = by the 1. part) Space EHFI.
 \square E. D.

PROP. LVII.

Fig.
XX.

IF, on either Asymptote be set off continual Proportionals, and, from their Extremities right Lines be drawn Parallel to the other Asymptote, then the Spaces between these Lines shall be as the Logarithms of the Ratio's of the Lines which bound them. That is, if $Ca, Cb, Cc, Cd, \&c.$ be \div then the Space $ackf$, is as the Logarithm of the Ratio of ck to af ; and the Space $adgf$, as the Logarithm of the Ratio of dg to af ; $\&c.$

DEMONST.

Let the Spaces between the Parallels be $A, B, C, D, \&c.$ (as in the Figure) then (by supposition) $\frac{Ca}{Cb} = \frac{Cb}{Cc}$, \therefore (by Prop. 56) $A = B$; and $\frac{Cc}{Cd} = \frac{Cd}{Ce}$, $\therefore B = C$, $\&c.$ that is, if $\frac{Ca}{Cb} = \frac{Cb}{Cc} = \frac{Cc}{Cd} = \frac{Cd}{Ce}$, $\&c.$ then $A = B = C = D$, $\&c.$ whence the Spaces are a Series of Continued Arithmetical Proportionals, fitted to a Series of Continued Geometrical Proportionals, and consequently the Addition of one Answers to the Multiplication of the other, which is the Property of Logarithms. as for Example.

Multiply

PART III. Of the HYPERBOLA. 99

Multiply the Geometrical Series $\frac{Ca}{Cb} = \frac{Cb}{Cc}$,
 the product will be $(\frac{Ca}{Cc} = \text{by Prop. 52.}) \frac{ck}{af}$,
 and add the Corresponding Arithmetical Series,
 and the Sum is $(A + B =)$ the Space $ackf$, Con-
 sequently the Space $ackf$, is as the Logarithm
 of the Ratio of ck to af . \mathcal{Q} . *E. D.*

PROP. LVIII.

THE Areas of two Hyperbolas having the
 same Transverse Axis, are as their Con-
 jugates.

DEMONST.

Let FB, fB , be two Hyperbolas described
 to the same Transverse Axis AB; then (by Prop.
 1.) GFq. BGA :: BDq. BCq, and Gfq. BGA
 :: Bdq. BCq. \therefore (by Equality) GFq. Gfq ::
 BDq. Bdq. and (by 22. E. 6.) GF. Gf :: BD.
 Bd. But the Sum of all the GF, Gf do respec-
 tively Constitute the Areas of the Hyperbolic
 Spaces BFG, BfG; therefore (by 12. E. 5.)
 those Areas are as the Conjugate Axes. \mathcal{Q} . *E. D.*

Fig.
XXI.

PROP. LIX.

Parallelograms Circumscribing any Diame-
 ters of an Hyperbola are equal.

DEMONST.

From the Vertex of the Diameter, and of the
 Curve, draw FI, BH, Parallel to the Asymp-
 tote

Fig.
XXII.

tote CQ, and to the other Asymptote let fall the Perpendiculars FG, BD, Put $IC = x$, $FI = y$, $BD = c$, and $CH = a$, then (from Similar Δ 's) $FP.FQ :: PI.IC$; but $PF = FQ \therefore PI = IC$ and $CP = 2x$, and FG (from the Similar Δ 's HBD, GIF, is) $= \frac{cy}{a}$, \therefore the Area of the Parallelogram PFCK $= PC \times FG = \frac{2cyx}{a} =$ (because, by *Scholium* to Prop. 52. $yx = a^2$), $2ac = CE \times BD = EBC$. Q.E.D.



Having



APPENDIX.

Having Considered the Properties of the *Parabola*, *Ellipse*, and *Hyperbola*, from their Construction in Plano, without any regard had to the *Cone*. I shall subjoin the Properties of the three Figures made by the cutting a *Cone* by a Plane; which Properties from their being the same with those before delivered, plainly prove the Figures whose Properties I have described, to be the true Sections of a *Cone*.

If a *Cone* be Cut by any Plane, to find the Figure of the Section.

*LET ABC, be a *Cone* standing on a Circular Base BC, and IEM its Section sought; and let KILM be any other Section Parallel to the Base, and meeting the former Section in HI; and ABC, a third Section, Perpendicularly Bisecting the former in EH and KL, and the *Cone* in the Triangle ABC, and producing EH (in Fig. 25.) till it meet AK in D; and having drawn EF and DG Parallel to KL, and meeting AB and AC in F, and G, call $EF = a$, $DG = b$, $ED = c$, the Abscissa $EH = x$, and the Ordinate $HI = y$; and by reason of the Simi-

* See Prop. 18. of Sir Isaac Newton's Algebra.

lar Triangles EHL, EDG, ED will be $DG :: EH. HL = \frac{bx}{c}$; then by reason of the Similar Triangles DEF, DHK, DE will be. $EF :: DH.$ ($c - x$ in Fig 24 and $c + x$ in Fig. 25) $HK = \frac{ac + ax}{c}$, lastly, since the Section KIL is Parallel to the Base, and consequently Circular, $HK \times HL$ will be $= HIq$, that is, $\frac{abcx + abx^2}{c^2} = y^2$; and if p be a fourth Proportional to c, a and b , then, $\frac{ab}{c}$ Equal p , and (by Substitution) $\frac{pcx + px^2}{c} = y^2$. and if X , and Y , be put for any other Abscissa, and Ordinate, then by the same reasoning it may be proved that $\frac{pcX + X^2}{c} = Y^2$. Hence,

1. When a *Cone* is cut by a Plane which intersects both its sides (as in Fig. 24.) then the Property of the Curve made by the Plane of that Section, will be such, that $c - x \times x. y^2 :: (c.p ::) c - X \times X. Y^2$; which is the same Property with Coroll. to Prop. 2. of the *Ellipse* foregoing.

2. When a Plane cuts the Base and side of a *Cone* continued from the Vertex, (as in Fig. 25.) the Property of the Curve made by the Plane of that Section, will be such, that $c + x \times x. y^2 :: (c.p ::) c + X \times X. Y^2$; which is the same with the preceding Coroll. to Prop. 2. of the *Hyperbola*.

If a *Cone* be cut by a Plane Parallel to one of its sides, (as in Fig. 26) and if, $AF = a$, $HK = b$, EH the Abscissa $= x$, and the Ordinate $IH = y$, then, (by reason of the Similar Triangles AFE , EHL) $AF \cdot (FE =) KH :: EH \cdot HL = \frac{bx}{a}$; but $KH \times HL = HI^2$, that is, $\frac{b^2 x}{a} = y^2$; and if you make p , a third Proportional to a , and b , then $\frac{b^2}{a} = p$, and (by Substitution) $px = y^2$; and if you put X , and Y , for any other Abscissa and Ordinate, then by the same manner of reasoning it may be proved that $pX = Y^2$. whence, $Y^2 : y^2 :: (pX \cdot px ::) X \cdot x$. which is the same with the preceding Cor. to Prop. 1. of the *Parabola*.



F I N I S.

A P E N D I X

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Hyperbola

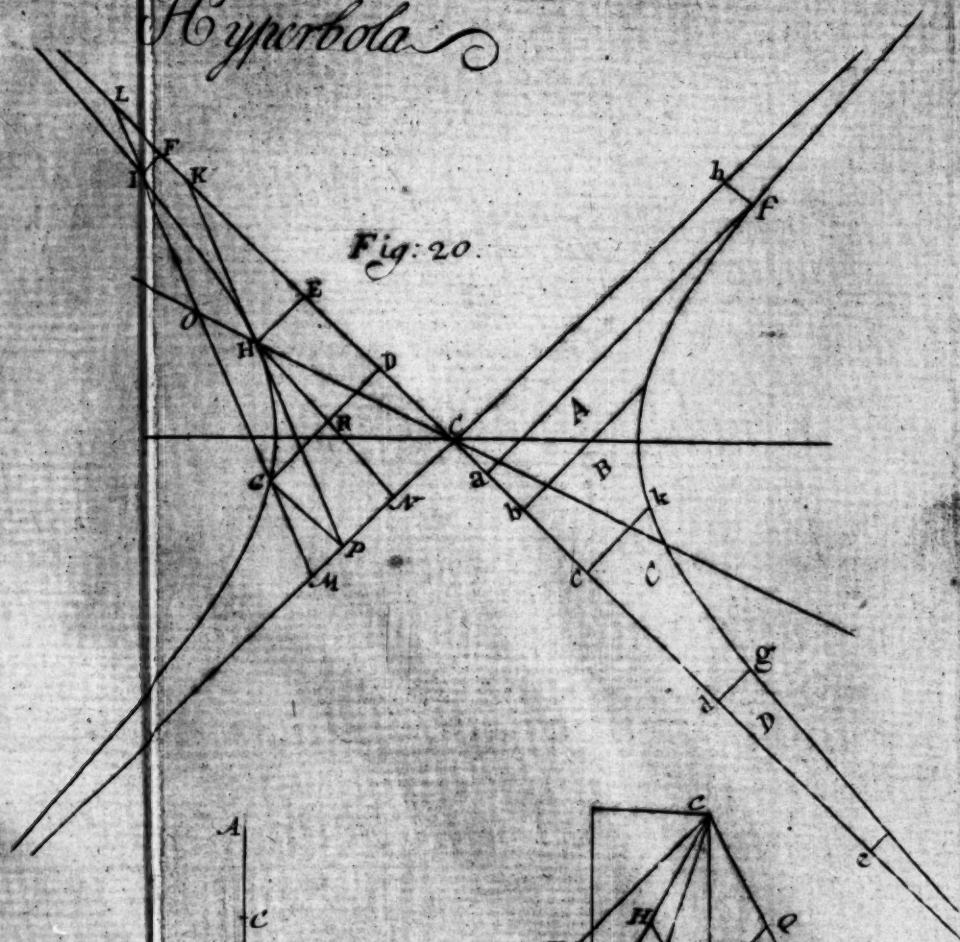


Fig. 20.

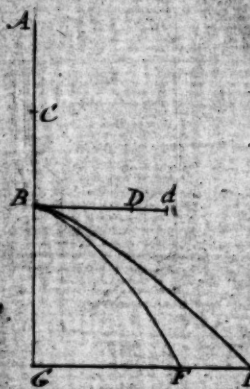


Fig. 21.

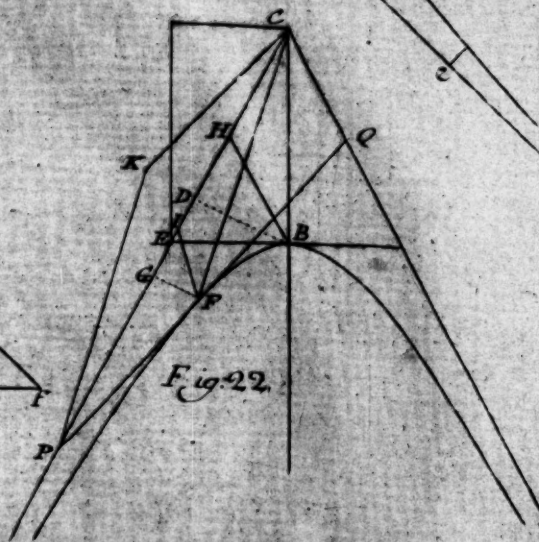


Fig. 22.